THE SIGNATURE OF FIBER BUNDLES

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Abstract. Let $Y \rightarrow Z \rightarrow X$ be a locally trivial fiber bundle in the category of oriented topological manifolds. It is shown that if the identity component of the structure group $G$ has finite index, then \((\text{signature of } Z) = (\text{signature of } X) \cdot (\text{signature of } Y)\).

Let $F \rightarrow E \rightarrow B$ be a locally trivial fiber bundle such that
1. $E$, $F$, $B$ are closed, oriented topological manifolds.
2. $E$, $F$, $B$ are coherently oriented, that is, the orientation of $F$ and $B$ determine that of $E$.

In this situation, does it follow that \(\sigma(E) = \sigma(B) \cdot \sigma(F)\), where \(\sigma(\ )\) denotes the signature homomorphism?

That additional conditions are necessary is shown both by Kodaira [4] and Atiyah [1] when they produce a locally trivial fibering of a complex surface by a complex surface such that the total space has a nonzero signature. In fact, in the smooth case, Atiyah produces a formula computing \(\sigma(E)\) and showing the dependency on the fundamental group of $B$.

The approach of this paper is to look at the structure group $G$ of the bundle and determine conditions on $G$ in order to obtain an affirmative answer to the above question. If $G$ is any topological group, let $\Gamma = G/G_0$, where $G_0$ is the connected component of the identity. The main result is the

**Theorem.** Let $G$ be a locally compact, finite dimensional topological group such that $|\Gamma|$ is finite. If $F \rightarrow E \rightarrow B$ is any oriented locally trivial topological fiber bundle with structure group $G$, then $\sigma E = \sigma B \cdot \sigma F$.

**Remark 1.** The theorem obviously remains valid if the structure group of the bundle is not, a fortiori, $G$, but can be reduced to $G$.

**Remark 2.** The hypothesis that $G$ be locally compact, finite dimensional only exists to insure that $G \rightarrow \Gamma$ possesses a local cross section. Any other hypothesis on $G$ insuring this is equally valid. See [3], for instance.

**Proposition 1.** If $\Gamma = (e)$, then $\sigma E = \sigma B \cdot \sigma F$.
Proof. This is essentially the theorem of Chern-Hirzebruch and Serre in [2]. One only needs to check that \( \pi_1(B) \) acts trivially on \( H^*(F) \). However this follows from the following two well-known facts.

(a) Let \( G \) be any group and \( B_G \) the classifying space for \( G \). If \( F \) is any left \( G \)-space, then the action \( \phi: \pi_1(B_G) \times H^*(F) \to H^*(F) \) in the bundle \( E_G \times_G F \to B_G \) is given by \( \phi(a, u) = y^*u \) where \( y \in G \) is any representative of \( \partial z \in \pi_0(G) \).

It follows that if \( G \) is connected, then the action is trivial.

(b) Let \((B, E', G, p')\) be the associated principal bundle to \( F \to E' \to B \), so that \( E = E' \times_G F \). Consider the following diagram.

\[
\begin{array}{ccc}
F & \to & E = E' \times_G F \to B \\
\downarrow & & \downarrow \\
\Gamma & = & \Gamma
\end{array}
\]

\[
\begin{array}{ccc}
F & \to & \tilde{E} = E' \times_G \tilde{F} \to E'/G_0 = \tilde{B} \\
\downarrow & & \downarrow \\
\Gamma & = & \Gamma
\end{array}
\]

Since \( G \to \Gamma \) has a local cross section, the columns are locally trivial fiber bundles with structure group \( G \) and fiber \( \Gamma \), where \( G \) acts on \( \Gamma \) via left translation. If we extend the structure group to \( \Gamma \), those columns become principal \( \Gamma \)-bundles. The middle row \( F \to \tilde{E} \to B \) is a \( G_0 \)-bundle.

Since \( B \) and \( F \) are manifolds and \( \Gamma \) is finite, it is obvious that all the spaces involved are closed manifolds. Therefore we may apply Proposition 1 to the middle row and conclude \( \sigma(E) = \sigma(B) \sigma(F) \).

Now since the two columns are principal \( \Gamma \)-bundles, i.e. finite covering spaces, it is clear that in order to prove the theorem it is sufficient to demonstrate the following result.

**Theorem.** Let \( X \) be a closed, connected, oriented topological manifold and \( \Gamma \) a finite group acting on \( X \) without fixed points and preserving the orientation. Then \( \sigma(X) = |\Gamma| \sigma(X/\Gamma) \).

Proof. In [6] there are constructed, for any oriented euclidian bundle \( \xi \) over a sufficiently nice space, rational Pontrjagin classes, or equivalently Hirzebruch classes, \( l(\xi) \). These classes satisfy naturality and Whitney formulas and are the rationalization of the ordinary Pontrjagin or Hirzebruch classes if \( \xi \) happens to be a vector bundle. Moreover if \( l(M) \) denotes \( l(\tau_M) \) for \( M \) an oriented topological manifold, then \( \langle l(M), [M] \rangle = \sigma(M) \).

Now if \( X \to X/\Gamma \) is a covering map, it is a local homeomorphism and so induces a map \( d\pi: \tau_X \to \tau_{X/\Gamma} \) which is an isomorphism on fibers i.e., \( \pi^*(\tau_{X/\Gamma}) = rX \). Therefore by naturality of the \( l \)-classes, \( \pi^*l(X/\Gamma) = l(X) \).
But since $X \to X/\Gamma$ is a finite covering, $\pi_* [X] = [\Gamma] \cdot [X/\Gamma]$. It follows that

$$
\sigma(X) = \langle l(X), [X] \rangle = \langle \pi^* l(X/\Gamma), [X] \rangle = \langle l(X/\Gamma), \pi_* [X] \rangle = [\Gamma] \cdot \sigma(X/\Gamma).
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REFERENCES


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