THE SIGNATURE OF FIBER BUNDLES

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Abstract. Let \( Y \hookrightarrow Z \twoheadrightarrow X \) be a locally trivial fiber bundle in the category of oriented topological manifolds. It is shown that if the identity component of the structure group \( G \) has finite index, then \( \text{signature of } Z = (\text{signature of } X) \cdot (\text{signature of } Y) \).

Let \( F \twoheadrightarrow E \twoheadrightarrow B \) be a locally trivial fiber bundle such that
(1) \( E, F, B \) are closed, oriented topological manifolds.
(2) \( E, F, B \) are coherently oriented, that is, the orientation of \( F \) and \( B \) determine that of \( E \).

In this situation, does it follow that \( \sigma(E) = \sigma(B) \cdot \sigma(F) \), where \( \sigma(\cdot) \) denotes the signature homomorphism?

That additional conditions are necessary is shown both by Kodaira [4] and Atiyah [1] when they produce a locally trivial fibering of a complex surface by a complex surface such that the total space has a nonzero signature. In fact, in the smooth case, Atiyah produces a formula computing \( \sigma(E) \) and showing the dependency on the fundamental group of \( B \).

The approach of this paper is to look at the structure group \( G \) of the bundle and determine conditions on \( G \) in order to obtain an affirmative answer to the above question. If \( G \) is any topological group, let \( \Gamma = G/G_0 \), where \( G_0 \) is the connected component of the identity. The main result is the

Theorem. Let \( G \) be a locally compact, finite dimensional topological group such that \( |\Gamma| \) is finite. If \( F \twoheadrightarrow E \twoheadrightarrow B \) is any oriented locally trivial topological fiber bundle with structure group \( G \), then \( \sigma E = \sigma B \cdot \sigma F \).

Remark 1. The theorem obviously remains valid if the structure group of the bundle is not, a fortiori, \( G \), but can be reduced to \( G \).

Remark 2. The hypothesis that \( G \) be locally compact, finite dimensional only exists to insure that \( G \twoheadrightarrow \Gamma \) possesses a local cross section. Any other hypothesis on \( G \) insuring this is equally valid. See [3], for instance.

Proposition 1. If \( \Gamma = (e) \), then \( \sigma E = \sigma B \cdot \sigma F \).
Proof. This is essentially the theorem of Chern-Hirzebruch and Serre in [2]. One only needs to check that $\pi_1(B)$ acts trivially on $H^*(F)$. However this follows from the following two well-known facts.

(a) Let $G$ be any group and $B_G$ the classifying space for $G$. If $F$ is any left $G$-space, then the action $\phi: \pi_1(B_G) \times H^*(F) \to H^*(F)$ in the bundle $E_G \times GF \to B_G$ is given by $\phi(g, u) = y^*(u)$ where $g \in G$ is any representative of $\partial g \pi_0(G)$.

It follows that if $G$ is connected, then the action is trivial.

(b) Let $(B', E', G, p')$ be the associated principal bundle to $F \to E \to B$, so that $E = E' \times_G F$. Consider the following diagram.

\[
\begin{array}{ccc}
F & \to & E = E' \times G F \overset{p}{\to} B \\
\downarrow & & \downarrow \\
F & \to & \tilde{E} = E' \times \tilde{G} F \overset{\tilde{p}}{\to} E'/G_0 = \tilde{B} \\
\uparrow & & \uparrow \\
\Gamma & \to & \Gamma
\end{array}
\]

Since $G \to \Gamma$ has a local cross section, the columns are locally trivial fiber bundles with structure group $G$ and fiber $\Gamma$, where $G$ acts on $\Gamma$ via left translation. If we extend the structure group to $\Gamma$, those columns become principal $\Gamma$ bundles. The middle row $F \to E \to B$ is a $G_0$-bundle.

Since $B$ and $F$ are manifolds and $\Gamma$ is finite, it is obvious that all the spaces involved are closed manifolds. Therefore we may apply Proposition 1 to the middle row and conclude $\sigma(E) = \sigma(B) \sigma(F)$.

Now since the two columns are principal $\Gamma$-bundles, i.e. finite covering spaces, it is clear that in order to prove the theorem it is sufficient to demonstrate the following result.

Theorem. Let $X$ be a closed, connected, oriented topological manifold and $\Gamma$ a finite group acting on $X$ without fixed points and preserving the orientation. Then $\sigma(X) = |\Gamma| \sigma(X/\Gamma)$.

Proof. In [6] there are constructed, for any oriented euclidian bundle $\xi$ over a sufficiently nice space, rational Pontrjagin classes, or equivalently Hirzebruch classes, $l(\xi)$. These classes satisfy naturality and Whitney formulas and are the rationalization of the ordinary Pontrjagin or Hirzebruch classes if $\xi$ happens to be a vector bundle. Moreover if $l(M)$ denotes $l(\tau_M)$ for $M$ an oriented topological manifold, then $\langle l(M), [M] \rangle = \sigma(M)$.

Now if $X \to X/\Gamma$ is a covering map, it is a local homeomorphism and so induces a map $d\pi: \tau X \to \tau X/\Gamma$ which is an isomorphism on fibers i.e., $\pi^*(\tau X/\Gamma) = rX$. Therefore by naturality of the $l$-classes, $\pi^*l(X/\Gamma) = l(X)$. 

\[
\begin{array}{ccc}
F & \to & E = E' \times G F \overset{p}{\to} B \\
\downarrow & & \downarrow \\
F & \to & \tilde{E} = E' \times \tilde{G} F \overset{\tilde{p}}{\to} E'/G_0 = \tilde{B} \\
\uparrow & & \uparrow \\
\Gamma & \to & \Gamma
\end{array}
\]
But since \( X \to X/\Gamma \) is a finite covering, \( \pi_* [X'] = [\Gamma] \cdot [X/\Gamma] \). It follows that

\[
\sigma(X) = \langle l(X), [X] \rangle = \langle \pi^* l(X/\Gamma), [X] \rangle \\
= \langle l(X/\Gamma), \pi_* [X] \rangle = [\Gamma] \cdot \sigma(X/\Gamma) .
\]

REFERENCES


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