Martín’s Axiom and Saturated Models

Erik Ellentuck and R. V. B. Rucker

Abstract. $2^{\aleph_0} > \aleph_1$ is consistent with the existence of an ultrafilter $F$ on $\omega$ such that for every countable structure $\mathfrak{A}$ the ultrapower $\mathfrak{A}^{\omega}/F$ is saturated.

1. Introduction. Let $\mathfrak{A}$ be an infinite structure with cardinality $\leq 2^{\aleph_0}$ and of countable length and let $\lambda$ be the least cardinal such that $2^{\aleph_0} < 2^\lambda$. Note that this makes $\lambda$ regular and $\aleph_1 \leq \lambda \leq 2^{\aleph_0}$.

(1) $\mathfrak{A}$ has a $\lambda$-saturated elementary extension $\mathfrak{B}$ of cardinality $2^{\aleph_0}$ (cf. [4]).

In general, even if $\mathfrak{A}$ is countable, we can neither reduce the cardinality of $\mathfrak{B}$ nor increase its degree of saturation. For the former, take $\mathfrak{A}$ to be the rational numbers with their usual ordering. If $\mathfrak{B}$ is $\aleph_1$-saturated, then $\mathfrak{B}$ is an $\eta_1$ set, but we know that the cardinality of every $\eta_1$ set is $\geq 2^{\aleph_0}$. For the latter, take $\mathfrak{A}$ to be $\langle \omega, +, \cdot \rangle$. If $\mathfrak{B}$ is $\lambda$-saturated, then there is a set of $\lambda$ primes $P \subseteq |\mathfrak{B}|$. But then by using $\lambda^+$-saturation we can find an injective map from the power set of $P$ into $|\mathfrak{B}|$, thus causing $\mathfrak{B}$ to have cardinality $\geq 2^\lambda > 2^{\aleph_0}$. For any cardinal $\kappa$ with $\aleph_0 \leq \kappa < \lambda$ we have

(2) There is a nonprincipal ultrafilter $F$ on $\kappa$ such that $\mathfrak{A}^{\kappa}/F$ is a $\kappa^+$-saturated elementary extension of $\mathfrak{A}$ having cardinality $2^{\aleph_0}$ (cf. [1], [2]).

Thus if $\kappa^+ = \lambda$, the $\mathfrak{B}$ of (1) can be taken as an ultrapower of $\mathfrak{A}$ with index set $\kappa$. That we cannot, in general, use a smaller index set is shown in Theorem 7. Let CH assert the continuum hypothesis. As special cases of (1), (2) we have

(1') If CH, then $\mathfrak{A}$ has a saturated elementary extension $\mathfrak{B}$ of cardinality $2^{\aleph_0}$.

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2 National Science Foundation Trainee.

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as well as

(2') If CH, then for every nonprincipal ultrafilter $F$ on $\omega$, $\mathcal{U}_{\omega}/F$ is a saturated elementary extension of $\mathcal{U}$ having cardinality $2^{\aleph_0}$ (cf. [1]).

In this paper we shall consider the analogues of (1') and (2') under the assumption of Martin's axiom MA.

2. Preliminaries. We work within ZFC (Zermelo-Fraenkel set theory with the axiom of choice). We identify an ordinal (number) with the set of its predecessors and let lower case Greek letters range over the ordinals. By a cardinal (number) we mean an initial ordinal and reserve $\kappa$, $\lambda$ for cardinals. If $\kappa$ is a cardinal, then $\kappa^+$ is the least cardinal $\geq \kappa$. For any function $f$ let $\delta(f) \subseteq \eta(y)$ denote the domain (range) of $f$. If $A$, $B$ are sets then $A^B$ is the set of all functions mapping $B$ into $A$, $S(A)$ is the power set of $A$, and $\text{Card}(A)$ is the cardinality of $A$, i.e. the unique cardinal bijective onto $A$. Let $\omega$ be the finite ordinals and for convenience we write $\mathcal{N} = \text{Card} S(\omega)$. Let $S_n(a) = \{ x \in S(A) \mid \text{Card}(x) < \kappa \}$, $S_n(A) = \{ x \in S(A) \mid \text{Card}(A - x) < \kappa \}$ where $A - x$ is the complement of $x$ in $A$, and $H_n(A, B) = \{ f \mid (\exists x \in S_n(A)) f \in B^x \}$. If $A \subseteq S(\omega)$ let

$$((A)) = \bigcap \{ F \mid A \subseteq F \subseteq S(\omega) \text{ and } F \text{ is an ultrafilter} \},$$

and say that $A$ satisfies the finite intersection property (finite intersection property) if $(\forall x \in S_n(A)) \bigcap x \neq \emptyset$. Finally let $S^*(A) = \{ x \in S(A) \mid \text{Card}(x) = \text{Card}(A) \}.$

Let $\mathcal{A} = (A, R_\xi)^{\xi < \alpha}$ be a relational structure. Let $|\mathcal{A}| = A$ and let the cardinality of $\mathcal{A}$ be $\text{Card}(A)$. We say $\mathcal{A}$ is countable if its cardinality is $\leq \omega$. We call $\alpha$ the length of $\mathcal{A}$ and let the type of $\mathcal{A}$ be that $t \in \omega^\alpha$ such that $t(\xi) = n$ if and only if $R_\xi$ is $n$-ary. $L(\mathcal{A})$ is the first order language with equality containing $\alpha$ predicates $\{ P_\xi \mid \xi < \alpha \}$ such that $P_\xi$ is $t(\xi)$-ary and is interpreted in $\mathcal{A}$ as $R_\xi$. For any language $L$ let $\text{Fm}(L)$ be the set of all formulas in $L$ with exactly one free variable $\nu_0$. We say $\Delta \subseteq \text{Fm}(L(\mathcal{A}))$ is satisfiable in $\mathcal{A}$ if there is an $x \in |\mathcal{A}|$ such that $x \in \Delta$. $\Delta$ is finitely satisfiable in $\mathcal{A}$ if each $\Gamma \in S_\omega(\Delta)$ is satisfiable in $\mathcal{A}$. Let $\kappa$ be the cardinality of $\mathcal{A}$ and let $\{ a_\xi \mid \xi < \kappa \}$ enumerate the elements of $|\mathcal{A}|$. If $\lambda$ is a cardinal, we say that $\mathcal{A}$ is $\lambda$-saturated if for each $\Delta \subseteq \text{Fm}(L(\mathcal{A}, a_\xi)^{\xi < \kappa})$ which involves fewer than $\lambda$ of the $a_\xi$ if $\Delta$ is finitely satisfiable in $\mathcal{A}(a_\xi)^{\xi < \kappa}$, then $\Delta$ is satisfiable in the same structure $\mathcal{A}$ is saturated if it is $\kappa$-saturated.

Let $\mathcal{P} = \langle P, \leq \rangle$ be a partially ordered set. We use the notion of [3] that $p \leq q$ means that $q$ contains more information than $p$, or that $q$ extends $p$. A set $D \subseteq P$ is dense if $(\forall p \in P) \ (\exists q \in D) \ p \leq q$. We say $p$ and $q$ are compatible if there is an $r \in P$ extending both $p$ and $q$. $\mathcal{P}$ satisfies the c.a.c.
(countable anti-chain condition) if every set of pairwise incompatible elements of $P$ is countable. Given a set $Z$ of dense subsets of $P$ we say that a set $G \subseteq P$ is $Z$-generic if (i) $q \in G$ and $p \leq q$ implies that $p \in G$, (ii) $p, q \in G$ implies that there is an $r \in G$ extending both $p$ and $q$, and (iii) $G \cap D \neq \emptyset$ for every $D \in Z$. MA (Martin’s axiom) is the following statement:

$$\text{If } \mathcal{P} \text{ is a partially ordered set satisfying the c.a.c. and if } Z \text{ is a set of less than } \aleph_0 \text{ dense subsets of } P \text{ then there exists a } Z\text{-generic subset of } P.$$  

It is not difficult to show that CH implies MA. More interesting are the results of [6] where it is shown that if ZFC is consistent then so is ZFC+$\text{MA}+\beth_1 = \aleph_2$. In fact $\beth_2$ can be replaced by many other possibilities, but we shall not go into this matter here. Due to this fact, the proof that a given statement is a consequence of MA also is a proof that the given statement is consistent with $\beth_1 > \beth_1$. Accordingly, if one has a particular aversion to MA, the results of this paper may still be viewed as valid consistency proofs.

3. The main result. Let $\lambda$ be the least cardinal such that $\aleph_0 < 2^\lambda$. Then

$$\text{If MA then } \lambda = \aleph_0 \text{ and } \aleph_0 \text{ is regular } \text{(cf. [3]).}$$

Thus we can conclude that for any $\mathfrak{A}$ as in §1

$$\text{If MA then } \mathfrak{A} \text{ has a saturated elementary extension } \mathfrak{B} \text{ of cardinality } \aleph_0.$$  

What about (2) under the assumption of MA? We have

**Theorem 1.** If MA then there exists a nonprincipal ultrafilter $F$ on $\omega$ such that for every countable structure $\mathfrak{A}$ with length $< \aleph_0$, the ultrapower $\mathfrak{A}^\omega/F$ is saturated.

Theorem 1 will follow as a corollary to a stronger result which we give as Theorem 2 below. But first we need some new notation. Let $\{\mathfrak{A}_i | i < \omega\}$ be a sequence of countable structures of the same type, and all of the same length $< \aleph_0$. Let $\{b_i | i < \aleph_0\}$ be an enumeration of $\prod_{i < \omega} \mathfrak{A}_i$ and let $\mathfrak{B} = \prod_{i < \omega} \mathfrak{A}_i, b_i \models \aleph_0, \mathfrak{B}(i) = (\mathfrak{A}_i, b_i(i)) \models \aleph_0$. If $F$ is an ultrafilter on $\omega$ and $\Delta \in \text{Fm}(\mathfrak{B})$ then define

$$\text{SH}_{\mathfrak{B}}(F, \Delta) \text{ if } (\forall \Gamma \in S_{\omega}(\Delta))(i \in \omega \mid \Gamma \text{ is satisfiable in } \mathfrak{B}(i)) \in F, \text{ and }$$

$$\text{SC}_{\mathfrak{B}}(F, \Delta) \text{ if } (\exists f \in |\mathfrak{B}|)(\forall \varphi \in \Delta)(i \in \omega \mid f(i) \text{ satisfies } \varphi \text{ in } \mathfrak{B}(i)) \in F.$$  

It is clear that the ultraproduct $\prod_{i < \omega} \mathfrak{A}_i/F$ is saturated if and only if $(\forall \Delta \in S_{\aleph_0}(\text{Fm}(\mathfrak{B})))\text{SH}_{\mathfrak{B}}(F, \Delta)$ implies $\text{SC}_{\mathfrak{B}}(F, \Delta)$. Then we have as the main result of our paper
Theorem 2. If MA then there exists a nonprincipal ultrafilter \( F \) on \( \omega \) such that for every sequence \( \{ \mathfrak{A}_i | i < \omega \} \) of countable structures of the same type and length \( < \mathfrak{K} \), the ultraproduct \( \prod_{i < \omega} \mathfrak{A}_i / F \) is saturated.

Proof. Without loss of generality we may assume that all countable structures have universe \( \omega \), for if the structure \( \mathfrak{A} \) is finite then we can find a structure \( \mathfrak{A}' \) with universe \( \omega \) and a unary predicate \( P(n) \) so that \( \mathfrak{A}' \) relativized to the denotation of \( P \) is isomorphic to \( \mathfrak{A} \). Fix an enumeration \( \{ b_i | i < \mathfrak{K} \} \) of \( \omega^\omega \) and an enumeration \( \{ \langle \mathfrak{B}_i, \Delta_i \rangle | i < \mathfrak{K} \} \) which contains every pair \( \langle \mathfrak{B}, \Delta \rangle \) such that \( \mathfrak{B} = \prod_{i < \omega} \mathfrak{A}_i, b_i / \mathfrak{B} \) for some \( \{ \mathfrak{A}_i | i < \omega \} \) as in the statement of the theorem and such that \( \Delta \in \mathfrak{S}(F(L(\mathfrak{B}))) \). Moreover assume that every \( \langle \mathfrak{B}, \Delta \rangle \) in the enumeration appears \( \mathfrak{K} \) times. Also let \( \{ X_i | i < \mathfrak{K} \} \) be an enumeration of \( S(\omega) \). Now we give an inductive definition of a sequence \( \{ F_\xi | i < \mathfrak{K} \} \) of proper filters on \( \omega \) which satisfy the conditions: (i) \( F_0 = S_\omega(\omega) \), (ii) \( F_\xi \subseteq F_{\xi+1} \), (iii) if \( \text{SH}_{\mathfrak{B}_\xi}(F_\xi, \Delta_\xi) \) then \( \text{SC}_{\mathfrak{B}_\xi}(F_{\xi+1}, \Delta_\xi) \), (iv) either \( X_\xi \in F_{\xi+1} \) or \( \omega - X_\xi \in F_{\xi+1} \), (v) \( F_\xi \) is generated by \( \text{Card}(\omega + \xi) \) of its elements, (vi) if \( \xi \) is a limit ordinal then \( F_\xi = \bigcup \{ F_\alpha | \alpha < \xi \} \). It is clear that such a sequence can be defined if we can show how to get \( F_{\xi+1} \) given \( \{ F_\alpha | \alpha < \xi \} \) satisfying (i)–(vi). Take \( G_\xi \subseteq F_\xi \) to be a set of \( \text{Card}(\omega + \xi) \) elements such that \( F_\xi = ((G_\xi)) \). We may assume that \( G_\xi \) is closed under finite intersections. The construction of \( F_{\xi+1} \) now splits into cases. Case 1: If not \( \text{SH}_{\mathfrak{B}_\xi}(F_\xi, \Delta_\xi) \) then we take \( F_{\xi+1} \) to be \( ((G_\xi \cup \{ \omega - X_\xi \})) \) if the latter satisfies the f.i.p. and to be \( ((G_\xi \cup \{ \omega - X_\xi \})) \) otherwise. Case 2: If \( \text{SH}_{\mathfrak{B}_\xi}(F_\xi, \Delta_\xi) \) then we must satisfy (iii) as well as (iv); it is here that we use MA. Let \( P = H_\omega(\omega, x) \times S_\omega(\Delta_\xi) \). For \( \langle p, K \rangle, \langle q, L \rangle \in P \) let \( \langle p, K \rangle \leq \langle q, L \rangle \) if (a) \( p \subseteq q \), (b) \( K \subseteq L \), and (c) if \( i \in \delta p - \delta q \) then \( q(i) \) satisfies \( K \) in \( \mathfrak{B}_\xi(i) \). If \( \mathfrak{P} = \langle P, \leq \rangle \) then \( \mathfrak{P} \) is a partially ordered set and satisfies the c.a.c. since any two elements of \( P \) with the same first component are compatible and there are only countably many first components. For \( \varphi \in \Delta_\xi \) let \( D_\varphi = \langle \{ p, K \in P | \varphi \in K \} \rangle \) and for \( Y \in G_\xi \) let \( E_Y = \langle \{ p, K \in P | p \not\in Y \}, Y \not\in \emptyset \rangle \). The \( D_\varphi \) are clearly dense subsets of \( P \). What we must show is that \( E_Y \) is dense for each \( Y \in G_\xi \). Consider any \( \langle p, K \rangle \in P \). By \( \text{SH}_{\mathfrak{B}_\xi}(F_\xi, \Delta_\xi) \) we have \( \{ \langle i \omega | K \text{ is satisfiable in } \mathfrak{B}_\xi(i) \} \in F_\xi \). Now every element of \( F_\xi \) is infinite because \( S_\omega(\omega) \subseteq F_\xi \) and hence we can find an element \( i \in Y - \delta p \) and an \( n \in \omega \) such that \( n \) satisfies \( K \) in \( \mathfrak{B}_\xi(i) \). Take \( q = p \cup \{ (i, n) \} \), i.e. \( q(i) = n \), and note that \( \langle p, K \rangle \leq \langle q, K \rangle \in E_Y \). Now we know that \( Z = \{ D_\varphi | \varphi \in \Delta_\xi \} \cup \{ E_Y | Y \in G_\xi \} \) is a set of less than \( \mathfrak{K} \) dense subsets of \( P \) and hence we may use MA to get a \( Z \)-generic \( H \subseteq P \). Let \( h_\xi = \bigcup \{ p | \exists K \langle p, K \rangle \in H \} \). Now \( G_\xi \cup \{ h_\xi \} \) satisfies the f.i.p. because \( H \) has a nonempty intersection with every \( E_Y \) for \( Y \in G_\xi \). Take \( F_{\xi+1} \) to be \( ((F_\xi \cup \{ h_\xi \}) \cup \{ X_\xi \}) \) if the latter satisfies the f.i.p. and to be \( ((F_\xi \cup \{ h_\xi \}) \cup \{ \omega - X_\xi \}) \) otherwise. We claim that \( \{ F_\alpha | \alpha < \xi + 1 \} \) satisfies
(i)–(vi), and note that this will follow from $SC_{B\mathcal{I}}(F_{\xi+1}, \Delta_\xi)$. Define a function $h_\xi: \omega \to \omega$ by $h_\xi(i) = h(i)$ if $i \in \delta h_\xi$ and to assume the value 0 otherwise. We claim that $(\forall \varphi \in \Delta_\xi) \{ i \in \omega | h_\xi(i) \text{ satisfies } \varphi \text{ in } B\mathcal{I}(i) \} \in F_{\xi+1}$. Consider some $\varphi \in \Delta_\xi$ and let $\langle p, K \rangle \in H \cap D_\varphi$. We will have proved our claim if we can show that $\delta h_\xi - \delta p \subseteq \{ i \in \omega | h_\xi(i) \text{ satisfies } \varphi \text{ in } B\mathcal{I}(i) \}$ because $\delta h_\xi - \delta p$ obviously belongs to $F_{\xi+1}$. Let $i \in \delta h_\xi - \delta p$. Then there is a $\langle q, L \rangle \in H$ such that $i \in \delta q$ as well as an $\langle r, M \rangle \in H$ extending both $\langle p, K \rangle$ and $\langle q, L \rangle$. Now $i \in \delta r - \delta p$ and hence $r(i)$ must satisfy $K$ in $B\mathcal{I}(i)$. But $r(i) = h_\xi(i) = h(i)$ and $\varphi \in K$ so that $h_\xi(i)$ satisfies $\varphi$ in $B\mathcal{I}(i)$. Thus we can now conclude that a sequence $\{ F_\xi | \xi < \mathfrak{K} \}$ of filters satisfying (i)–(vi) can be defined. Let $F = \bigcup \{ F_\xi | \xi < \mathfrak{K} \}$. We claim that $F$ satisfies the conclusion of our theorem. First, it is clear from the construction that $F$ is a proper nonprincipal ultrafilter. Now consider any sequence $\{ \mathcal{U}_i | i < \omega \}$ as in the hypothesis of our theorem and let $\mathcal{B} = \prod_{i < \omega} \mathcal{U}_i$, $b_\xi < \mathfrak{K}$ and $\Delta \in S_\mathcal{B}(L(\mathcal{B}))$. Suppose that $SH_{\mathfrak{B}}(F, \Delta)$. This says that a certain collection of fewer than $\mathfrak{K}$ subsets of $\omega$ is a subset of $F$. Then by the regularity of $\mathfrak{K}$ we can find an $\alpha < \mathfrak{K}$ such that $SH_{\mathfrak{B}}(F_\xi, \Delta)$ for every $\xi > \alpha$. Choose $\xi > \alpha$ such that $(\mathcal{B}_\xi, \Delta_\xi) = (\mathcal{B}, \Delta)$. But then we have $SC_{B\mathcal{I}}(F_{\xi+1}, \Delta_\xi)$ and hence $SC_{\mathfrak{B}}(F, \Delta)$. Q.E.D.

4. Some additions. By a theorem of Tarski we know that there are $\mathfrak{D}$ ultrafilters on $\omega$, where $\mathfrak{D} = \text{Card } S(\mathfrak{K})$. How many of them satisfy Theorem 2?

**Theorem 3.** If MA then there are $\mathfrak{D}$ nonprincipal ultrafilters $F$ on $\omega$ such that for every sequence $\{ \mathcal{U}_i | i < \omega \}$ of countable structures of the same type and length $< \mathfrak{K}$ the ultraproduct $\prod_{i < \omega} \mathcal{U}_i / F$ is saturated.

In order to prove Theorem 3 we need

**Lemma 4.** If MA then for any $F \subseteq S^*(\omega)$ with $\text{Card}(F) < \mathfrak{K}$ there exists an $X \in S^*(\omega)$ such that if $F$ is a proper filter then so is $((F \cup \{ X \}))$ and $((F \cup \{ \omega - X \}))$.

**Proof.** Let $P = H_\omega(\omega, \{ 0, 1 \})$ and for $p, q \in P$ put $p \leq q$ if $p \sqsubseteq q$. Clearly $\mathcal{P} = \langle P, \leq \rangle$ is a partially ordered set that satisfies the c.a.c. For each $Y \in F$ define $D_Y = \{ p \in P | (\exists i, j \in \delta p \cap Y) \ p(i) = 0 \text{ and } p(j) = 1 \}$. Since each $Y \in F$ is infinite $D_Y$ is dense. Hence $Z = \{ D_Y | Y \in F \}$ is a collection of less than $\mathfrak{K}$ dense subsets of $P$. Now use MA to get a $Z$-generic $H \subseteq P$ and let $h = \bigcup H$. Then clearly $X = \{ i \in \omega | h(i) = 1 \}$ satisfies the lemma. Q.E.D.

**Proof of Theorem 3.** Use Lemma 4 to get a map $r$ from $S_\mathcal{B}(S^*(\omega))$ into $S^*(\omega)$ such that $r(F)$ is some $X$ which satisfies the conclusion of Lemma 4 for $F$. Let $A = \{ \xi < \mathfrak{K} | \xi \text{ is a limit ordinal} \}$. Clearly $\text{Card}(2^A) = \mathfrak{D}$. 


We will show that corresponding to each \( f \in 2^A \) there is a distinct \( F_f \) satisfying Theorem 2. Consider some \( f \in 2^A \). Run through the construction in the proof of Theorem 2 as before, only this time replace condition (vi) by the new condition (vi): if \( \xi \) is a limit ordinal take \( F_\xi \) to be \( (\{ K_\xi \cup \{ r(K_\xi) \} \} ) \) if \( f(\xi) = 1 \) and to be \( (\{ K_\xi \cup \{ \omega - r(K_\xi) \} \} ) \) otherwise, where \( K_\xi = \bigcup \{ F_\alpha \mid \alpha < \xi \} \). Let \( F_f \) be the resulting ultrafilter. It is easy to verify that if \( f, g \in 2^A \) and \( f \neq g \) then \( F_f \) and \( F_g \) are distinct ultrafilters satisfying Theorem 2. Q.E.D.

An interesting consequence of Lemma 4 is

**Corollary 5.** If MA then no nonprincipal proper ultrafilter on \( \omega \) is generated by fewer than \( \aleph_1 \) of its members.

That Corollary 5 need not hold in the absence of MA is pointed out in [2].

Can Theorem 1 be extended, in analogy with (2'), to all \( \mathfrak{A} \) of length \( < \aleph_1 \) and cardinality \( \leq \aleph_0 \)?

**Theorem 6.** If \( \mathfrak{K} > \aleph_1 \) then for any nonprincipal ultrafilter \( F \) on \( \omega \) the ultrapower \( \mathfrak{B} = \langle \mathfrak{K}_1, \leq \rangle^\omega \mathfrak{A}/F \) is not saturated.

**Proof.** The cardinality of \( \mathfrak{B} \) is \( \geq \aleph_1 > \aleph_0 \) so that it will be enough to show that \( \mathfrak{B} \) is not \( \aleph_0 \)-saturated. For each \( \xi \in \mathfrak{K}_1 \) let \( b_\xi \) be the function with domain \( \omega \) and range \( \{ \xi \} \). Let \( \mathfrak{B}' = \langle \mathfrak{B}, b_\xi \rangle_{\xi < \aleph_1} \) and let \( c_{\xi} \) be a constant in \( L(\mathfrak{B}') \) denoting \( b_\xi \). For \( \Delta \) take \( \{ c_{\xi} \leq c_{\xi_0} \mid \xi \in \mathfrak{K}_1 \} \) and notice that \( SH_{\mathfrak{B}'}(F, \Delta) \) is true. We claim that \( SC_{\mathfrak{B}'}(F, \Delta) \) is false. For otherwise let \( f \in \mathfrak{K}_1^\omega \) satisfy \( \Delta \) in \( \mathfrak{B}' \). Define \( S_\xi = \{ i \in \omega \mid b_\xi(i) \leq f(i) \} \) and notice that \( S_\xi \subseteq S_\eta \) for \( \xi < \eta < \aleph_1 \) and \( S_\xi \in F \) for every \( \xi < \aleph_1 \). Thus there is a \( \xi_0 \) such that \( S_{\xi_0} = S_\xi \) for \( \xi < \xi_0 < \xi_1 \). But then if \( i \in S_{\xi_0} \) we must have \( i \in \bigcap \{ S_\xi \mid \xi < \aleph_1 \} \) which implies that \( \mathfrak{K}_1 \leq f(i) \), a contradiction. Q.E.D.

Thus the limitation on the cardinality of \( \mathfrak{A} \) in Theorem 1 cannot be removed. Finally let us return to the claim made just after (2). Let \( \lambda \) be the least cardinal such that \( \aleph_0 < \text{Card} \ S(\lambda) \). If MA then by Theorem 1 and (4) there will be a nonprincipal ultrafilter \( F \) on \( \omega \) such that \( \mathfrak{A} \) is a countably infinite structure of countable length then \( \mathfrak{A}^\omega \mathfrak{A}/F \) is a \( \lambda \)-saturated elementary extension of \( \mathfrak{A} \) having cardinality \( \aleph_0 \). Now the index \( \omega \) of this ultrapower is very small. In the next theorem we show (in the absence of MA) that it is consistent that for no \( \kappa \) with \( \kappa^+ < \lambda \) and for no nonprincipal ultrafilter \( F \) on \( \kappa \) is \( \mathfrak{A}^\kappa \mathfrak{A}/F \) \( \lambda \)-saturated, i.e. that the index of the ultrapower must be as large as possible in order to get \( \lambda \)-saturation.

**Theorem 7.** There is a Boolean valued model \( V^\mathfrak{A} \) of set theory in which \( \lambda = \aleph_1 = \aleph_2 \) and \( \mathfrak{B} = \langle \omega, \leq \rangle^\omega \mathfrak{A}/F \) is \( \lambda \)-saturated for no nonprincipal ultrafilter \( F \) on \( \omega \).
Proof. Most of our argument is reconstructed from [5]. Without loss of generality we may assume that $V=L$. Let $X=\{0, 1\}^\mathbb{N}$ with a topology generated by the subbasic open sets $B_{\xi n} = \{ f \in X \mid f(\xi) = n \}$ and let $\mu$ be a product measure on the Borel sets of $X$ which is induced by the unbiased measure on $\{0, 1\}$. Our complete Boolean algebra $\mathcal{B}$ is the quotient of the Borel sets of $X$ mod those Borel sets of measure 0. Cardinals are preserved when extending $V$ to $V^\mathcal{B}$ because $\mathcal{B}$ satisfies the c.a.c. and $V^\mathcal{B}$ satisfies $\lambda = \aleph = \aleph_2$ by the usual argument. Since $\mathcal{B}$ is a measure algebra it satisfies the $(\omega, \omega)$-weak distributive law

$$\prod_{n \in \omega} \sum_{m \in \omega} b_{nm} = \sum_{n \in \omega} \prod_{m \in \omega} \sum_{n \leq s(n)} b_{nm}$$

where $b : \omega \times \omega \to \mathcal{B}$, and consequently

$$\| f : \omega \to \omega \| = \prod_{n \in \omega} \sum_{m \in \omega} \| f(\xi) = m \| = \sum_{n \in \omega} \prod_{m \in \omega} \sum_{n \leq s(n)} \| f(\xi) = m \| = \| (\exists \xi \in \omega^\omega) (\forall n \in \omega) f(n) \leq s(n) \|.$$

Details can be found in [5]. Thus in $V^\mathcal{B}$ there is a set $G$ of $\aleph_1$ functions $g \in \omega^\omega \cap V$ which dominate every one of the $\aleph_2$ functions $f \in \omega^\omega$ in the sense that $f < g$ if $(\forall n \in \omega) f(n) \leq g(n)$. Then working in $V^\mathcal{B}$, let $\{ b_\xi \mid \xi < \aleph_1 \}$ enumerate $G$, let $\mathcal{B}' = (\mathcal{B}, b_\xi < \aleph_1)$, and let $c_\xi$ be a constant in $L(\mathcal{B}')$ denoting $b_\xi$. For $\Delta$ take $\{ c_\xi \mid \xi \in \aleph_1 \}$ and notice that $S(\mathcal{B}'(F, \Delta)$ is false because otherwise there would be a function $f$ which is dominated by no $g \in G$. Q.E.D.

References


School of Mathematics, The Institute for Advanced Study, Princeton, New Jersey 08540

Department of Mathematics, Rutgers, The State University, New Brunswick, New Jersey 08903