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# MARTIN'S AXIOM AND SATURATED MODELS

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ABSTRACT. $2^{\aleph_0}>\aleph_1$ is consistent with the existence of an ultrafilter $F$ on $\omega$ such that for every countable structure $\mathfrak{A}$ the ultrapower $\mathfrak{A}^\omega/F$ is saturated.

1. **Introduction.** Let $\mathfrak{A}$ be an infinite structure with cardinality $\leqq 2^{\aleph_0}$ and of countable length and let $\lambda$ be the least cardinal such that $2^{\aleph_0}<2^\lambda$. Note that this makes $\lambda$ regular and $\aleph_1\leqq\lambda\leqq 2^{\aleph_0}$.

(1) $\mathfrak{A}$ has a $\lambda$-saturated elementary extension $\mathfrak{B}$ of cardinality $2^{\aleph_0}$ (cf. [**4**]).

In general, even if $\mathfrak{A}$ is countable, we can neither reduce the cardinality of $\mathfrak{B}$ nor increase its degree of saturation. For the former, take $\mathfrak{A}$ to be the rational numbers with their usual ordering. If $\mathfrak{B}$ is $\aleph_1$-saturated, then $\mathfrak{B}$ is an $\eta_1$ set, but we know that the cardinality of every $\eta_1$ set is $\geqq 2^{\aleph_0}$. For the latter, take $\mathfrak{A}$ to be $\langle\omega, +, \cdot\rangle$. If $\mathfrak{B}$ is $\lambda$-saturated, then there is a set of $\lambda$ primes $P\subseteq|\mathfrak{B}|$. But then by using $\lambda^+$-saturation we can find an injective map from the power set of $P$ into $|\mathfrak{B}|$, thus causing $\mathfrak{B}$ to have cardinality $\geqq 2^\lambda>2^{\aleph_0}$. For any cardinal $\kappa$ with $\aleph_0\leqq\kappa<\lambda$ we have

(2) There is a nonprincipal ultrafilter $F$ on $\kappa$ such that $\mathfrak{A}^\kappa/F$ is a $\kappa^+$-saturated elementary extension of $\mathfrak{A}$ having cardinality $2^{\aleph_0}$ (cf. [**1**], [**2**]).

Thus if $\kappa^+=\lambda$, the $\mathfrak{B}$ of (1) can be taken as an ultraproduct of $\mathfrak{A}$ with index set $\kappa$. That we cannot, in general, use a smaller index set is shown in Theorem 7. Let CH assert the continuum hypothesis. As special cases of (1), (2) we have

(1′) If CH, then $\mathfrak{A}$ has a saturated elementary extension $\mathfrak{B}$ of cardinality $2^{\aleph_0}$

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as well as

(2′) If CH, then for every nonprincipal ultrafilter $F$ on $\omega$, $\mathfrak{A}^{\omega}/F$ is a saturated elementary extension of $\mathfrak{A}$ having cardinality $2^{\aleph_0}$ (cf. [1]).

In this paper we shall consider the analogues of (1′) and (2′) under the assumption of Martin's axiom MA.

2. **Preliminaries.** We work within ZFC (Zermelo-Fraenkel set theory with the axiom of choice). We identify an *ordinal* (number) with the set of its predecessors and let lower case Greek letters range over the ordinals. By a *cardinal* (number) we mean an initial ordinal and reserve $\kappa$, $\lambda$ for cardinals. If $\kappa$ is a cardinal, then $\kappa^+$ is the least cardinal $>\kappa$. For any function $f$ let $\delta f\,(\rho f)$ denote the domain (range) of $f$. If $A$, $B$ are sets then $A^B$ is the set of all functions mapping $B$ into $A$, $S(A)$ is the power set of $A$, and $\mathrm{Card}(A)$ is the cardinality of $A$, i.e. the unique cardinal bijectible onto $A$. Let $\omega$ be the finite ordinals and for convenience we write $\aleph=\mathrm{Card}\,S(\omega)$. Let $S_\kappa(A)=\{x \in S(A) | \mathrm{Card}(x)<\kappa\}$, $\bar{S}_\kappa(A)=\{x \in S(A) | \mathrm{Card}(A-x)<\kappa\}$ where $A-x$ is the complement of $x$ in $A$, and $H_\kappa(A, B)=\{f | (\exists x \in S_\kappa(A)) f \in B^x\}$. If $A \subseteq S(\omega)$ let

$$((A)) = \bigcap\{F \mid A \subseteq F \subseteq S(\omega) \text{ and } F \text{ is an ultrafilter}\},$$

and say that $A$ satisfies the *f.i.p.* (*finite intersection property*) if $(\forall x \in S_\omega(\mathrm{A})) \bigcap x \neq \varnothing$. Finally let $S^*(A)=\{x \in S(A) | \mathrm{Card}(x)=\mathrm{Card}(A)\}$.

Let $\mathfrak{A}=\langle A, R_\xi\rangle_{\xi<\alpha}$ be a relational structure. Let $|\mathfrak{A}|=A$ and let the *cardinality* of $\mathfrak{A}$ be $\mathrm{Card}(A)$. We say $\mathfrak{A}$ is *countable* if its cardinality is $\leqq\omega$. We call $\alpha$ the *length* of $\mathfrak{A}$ and let the *type* of $\mathfrak{A}$ be that $t \in \omega^\alpha$ such that $t(\xi)=n$ if and only if $R_\xi$ is $n$-ary. $L(\mathfrak{A})$ is the first order language with equality containing $\alpha$ predicates $\{P_\xi | \xi<\alpha\}$ such that $P_\xi$ is $t(\xi)$-ary and is interpreted in $\mathfrak{A}$ as $R_\xi$. For any language $L$ let $\mathrm{Fm}(L)$ be the set of all formulas in $L$ with exactly one free variable $v_0$. We say $\Delta \subseteq \mathrm{Fm}(L(\mathfrak{A}))$ is *satisfiable* in $\mathfrak{A}$ if there is an $x \in |\mathfrak{A}|$ such that $x$ satisfies $\varphi$ in $\mathfrak{A}$ for every $\varphi \in \Delta$. $\Delta$ is *finitely satisfiable* in $\mathfrak{A}$ if each $\Gamma \in S_\omega(\Delta)$ is satisfiable in $\mathfrak{A}$. Let $\kappa$ be the cardinality of $\mathfrak{A}$ and let $\{a_\xi | \xi<\kappa\}$ enumerate the elements of $|\mathfrak{A}|$. If $\lambda$ is a cardinal, we say that $\mathfrak{A}$ is *$\lambda$-saturated* if for each $\Delta \subseteq \mathrm{Fm}(L(\langle\mathfrak{A}, a_\xi\rangle_{\xi<\kappa}))$ which involves fewer than $\lambda$ of the $a_\xi$ if $\Delta$ is finitely satisfiable in $\langle\mathfrak{A}, a_\xi\rangle_{\xi<\kappa}$, then $\Delta$ is satisfiable in the same structure. $\mathfrak{A}$ is *saturated* if it is $\kappa$-saturated.

Let $\mathscr{P}=\langle P, \leqq\rangle$ be a partially ordered set. We use the notion of [3] that $p\leqq q$ means that $q$ contains more information than $p$, or that $q$ *extends* $p$. A set $D \subseteq P$ is *dense* if $(\forall p \in P)\,(\exists q \in D)\, p\leqq q$. We say $p$ and $q$ are *compatible* if there is an $r \in P$ extending both $p$ and $q$. $\mathscr{P}$ satisfies the *c.a.c.*

(*countable anti-chain condition*) if every set of pairwise incompatible elements of $P$ is countable. Given a set $Z$ of dense subsets of $P$ we say that a set $G \subseteq P$ is *Z-generic* if (i) $q \in G$ and $p \leqq q$ implies that $p \in G$, (ii) $p, q \in G$ implies that there is an $r \in G$ extending both $p$ and $q$, and (iii) $G \cap D \neq \varnothing$ for every $D \in Z$. MA (Martin's axiom) is the following statement:

(3) If $\mathscr{P}$ is a partially ordered set satisfying the c.a.c. and if $Z$ is a set of less than $\aleph$ dense subsets of $P$ then there exists a $Z$-generic subset of $P$.

It is not difficult to show that CH implies MA. More interesting are the results of **[6]** where it is shown that if ZFC is consistent then so is ZFC$+$MA$+\aleph=\aleph_2$. In fact $\aleph_2$ can be replaced by many other possibilities, but we shall not go into this matter here. Due to this fact, the proof that a given statement is a consequence of MA also is a proof that the given statement is consistent with $\aleph>\aleph_1$. Accordingly, if one has a particular aversion to MA, the results of this paper may still be viewed as valid consistency proofs.

3. **The main result.** Let $\lambda$ be the least cardinal such that $\aleph<2^\lambda$. Then

(4) If MA then $\lambda = \aleph$ and $\aleph$ is regular (cf. **[3]**).

Thus we can conclude that for any $\mathfrak{A}$ as in §1

(1″) If MA then $\mathfrak{A}$ has a saturated elementary extension $\mathfrak{B}$ of cardinality $\aleph$.

What about (2) under the assumption of MA? We have

THEOREM 1. *If* MA *then there exists a nonprincipal ultrafilter $F$ on $\omega$ such that for every countable structure $\mathfrak{A}$ with length $<\aleph$, the ultrapower $\mathfrak{A}^\omega/F$ is saturated.*

Theorem 1 will follow as a corollary to a stronger result which we give as Theorem 2 below. But first we need some new notation. Let $\{\mathfrak{A}_i \mid i<\omega\}$ be a sequence of countable structures of the same type, and all of the same length $\alpha<\aleph$. Let $\{b_\xi \mid \xi<\aleph\}$ be an enumeration of $\prod_{i<\omega} |\mathfrak{A}_i|$ and let $\mathfrak{B}=\langle \prod_{i<\omega} \mathfrak{A}_i, b_\xi\rangle_{\xi<\aleph}$, $\mathfrak{B}(i)=\langle \mathfrak{A}_i, b_\xi(i)\rangle_{\xi<\aleph}$. If $F$ is an ultrafilter on $\omega$ and $\Delta \in \mathrm{Fm}(L(\mathfrak{B}))$ then define

$$\mathrm{SH}_{\mathfrak{B}}(F, \Delta) \text{ if } (\forall \Gamma \in S_\omega(\Delta))\{i \in \omega \mid \Gamma \text{ is satisfiable in } \mathfrak{B}(i)\} \in F,$$

$$\mathrm{SC}_{\mathfrak{B}}(F, \Delta) \text{ if } (\exists f \in |\mathfrak{B}|)(\forall \varphi \in \Delta)\{i \in \omega \mid f(i) \text{ satisfies } \varphi \text{ in } \mathfrak{B}(i)\} \in F.$$

It is clear that the ultraproduct $\prod_{i<\omega} \mathfrak{A}_i/F$ is saturated if and only if $(\forall \Delta \in S_\aleph(\mathrm{Fm}(L(\mathfrak{B}))))\mathrm{SH}_{\mathfrak{B}}(F, \Delta)$ implies $\mathrm{SC}_{\mathfrak{B}}(F, \Delta)$. Then we have as the main result of our paper

THEOREM 2. *If* MA *then there exists a nonprincipal ultrafilter $F$ on $\omega$ such that for every sequence $\{\mathfrak{A}_i | i<\omega\}$ of countable structures of the same type and length $<\aleph$, the ultraproduct $\prod_{i<\omega} \mathfrak{A}_i/F$ is saturated.*

PROOF. Without loss of generality we may assume that all countable structures have universe $\omega$, for if the structure $\mathfrak{A}$ is finite then we can find a structure $\mathfrak{A}'$ with universe $\omega$ and a unary predicate $P(v_0)$ so that $\mathfrak{A}'$ relativized to the denotation of $P$ is isomorphic to $\mathfrak{A}$. Fix an enumeration $\{b_\xi | \xi<\aleph\}$ of $\omega^\omega$ and an enumeration $\{\langle\mathfrak{B}_\xi, \Delta_\xi\rangle | \xi<\aleph\}$ which contains every pair $\langle\mathfrak{B}, \Delta\rangle$ such that $\mathfrak{B}=\langle\prod_{i<\omega}\mathfrak{A}_i, b_\xi\rangle_{\xi<\aleph}$ for some $\{\mathfrak{A}_i | i<\omega\}$ as in the statement of the theorem and such that $\Delta \in S_\aleph(F(L(\mathfrak{B})))$. Moreover assume that every $\langle\mathfrak{B}, \Delta\rangle$ in the enumeration appears $\aleph$ times. Also let $\{X_\xi | \xi<\aleph\}$ be an enumeration of $S(\omega)$. Now we give an inductive definition of a sequence $\{F_\xi | \xi<\aleph\}$ of proper filters on $\omega$ which satisfy the conditions: (i) $F_0=\bar{S}_\omega(\omega)$, (ii) $F_\xi\subseteq F_{\xi+1}$, (iii) if $\mathrm{SH}_{\mathfrak{B}_\xi}(F_\xi, \Delta_\xi)$ then $\mathrm{SC}_{\mathfrak{B}_\xi}(F_{\xi+1}, \Delta_\xi)$, (iv) either $X_\xi \in F_{\xi+1}$ or $\omega-X_\xi \in F_{\xi+1}$, (v) $F_\xi$ is generated by $\mathrm{Card}(\omega+\xi)$ of its elements, (vi) if $\xi$ is a limit ordinal then $F_\xi=\bigcup\{F_\alpha | \alpha<\xi\}$. It is clear that such a sequence can be defined if we can show how to get $F_{\xi+1}$ given $\{F_\alpha | \alpha\leqq\xi\}$ satisfying (i)–(vi). Take $G_\xi\subseteq F_\xi$ to be a set of $\mathrm{Card}(\omega+\xi)$ elements such that $F_\xi=((G_\xi))$. We may assume that $G_\xi$ is closed under finite intersections. The construction of $F_{\xi+1}$ now splits into cases. *Case* 1: If not $\mathrm{SH}_{\mathfrak{B}_\xi}(F_\xi, \Delta_\xi)$ then we take $F_{\xi+1}$ to be $((G_\xi\cup\{X_\xi\}))$ if the latter satisfies the f.i.p. and to be $((G_\xi\cup\{\omega-X_\xi\}))$ otherwise. *Case* 2: If $\mathrm{SH}_{\mathfrak{B}_\xi}(F_\xi, \Delta_\xi)$ then we must satisfy (iii) as well as (iv); it is here that we use MA. Let $P=H_\omega(\omega, \omega)\times S_\omega(\Delta_\xi)$. For $\langle p, K\rangle$, $\langle q, L\rangle \in P$ let $\langle p, K\rangle\leqq\langle q, L\rangle$ if (a) $p\subseteq q$, (b) $K\subseteq L$, and (c) if $i \in \delta q-\delta p$ then $q(i)$ satisfies $K$ in $\mathfrak{B}_\xi(i)$. If $\mathscr{P}=\langle P, \leqq\rangle$ then $\mathscr{P}$ is a partially ordered set and satisfies the c.a.c. since any two elements of $P$ with the same first component are compatible and there are only countably many first components. For $\varphi \in \Delta_\xi$ let $D_\varphi=\{\langle p, K\rangle \in P | \varphi \in K\}$ and for $Y \in G_\xi$ let $E_Y=\{\langle p, K\rangle \in P | \delta p\cap Y\neq\varnothing\}$. The $D_\varphi$ are clearly dense subsets of $P$. What we must show is that $E_Y$ is dense for each $Y \in G_\xi$. Consider any $\langle p, K\rangle \in P$. By $\mathrm{SH}_{\mathfrak{B}_\xi}(F_\xi, \Delta_\xi)$ we have $\{i \in \omega | K \text{ is satisfiable in } \mathfrak{B}_\xi(i)\} \in F_\xi$. Now every element of $F_\xi$ is infinite because $\bar{S}_\omega(\omega)\subseteq F_\xi$ and hence we can find an element $i \in Y-\delta p$ and an $n \in \omega$ such that $n$ satisfies $K$ in $\mathfrak{B}_\xi(i)$. Take $q=p\cup\{\langle i, n\rangle\}$, i.e. $q(i)=n$, and note that $\langle p, K\rangle\leqq\langle q, K\rangle \in E_Y$. Now we know that $Z=\{D_\varphi | \varphi \in \Delta_\xi\}\cup\{E_Y | Y \in G_\xi\}$ is a set of less than $\aleph$ dense subsets of $P$ and hence we may use MA to get a $Z$-generic $H\subseteq P$. Let $h_\xi=\bigcup\{p | (\exists K)\langle p, K\rangle \in H\}$. Now $G_\xi\cup\{\delta h_\xi\}$ satisfies the f.i.p. because $H$ has a nonempty intersection with every $E_Y$ for $Y \in G_\xi$. Take $F_{\xi+1}$ to be $((F_\xi\cup\{\delta h_\xi\}\cup\{X_\xi\}))$ if the latter satisfies the f.i.p. and to be $((F_\xi\cup\{\delta h_\xi\}\cup\{\omega-X_\xi\}))$ otherwise. We claim that $\{F_\alpha | \alpha\leqq\xi+1\}$ satisfies (i)–(vi), and note that this will follow from $SC_{\mathfrak{B}_\xi}(F_{\xi+1}, \Delta_\xi)$. Define a function $\bar{h}_\xi:\omega\to\omega$ by $\bar{h}_\xi(i)=h_\xi(i)$ if $i\in\delta h_\xi$ and to assume the value 0 otherwise. We claim that $(\forall\varphi\in\Delta_\xi)\,\{i\in\omega|\bar{h}_\xi(i)$ satisfies $\varphi$ in $\mathfrak{B}_\xi(i)\}\in F_{\xi+1}$. Consider some $\varphi\in\Delta_\xi$ and let $\langle p, K\rangle\in H\cap D_\varphi$. We will have proved our claim if we can show that $\delta h_\xi-\delta p\subseteq\{i\in\omega|\bar{h}_\xi(i)$ satisfies $\varphi$ in $\mathfrak{B}_\xi(i)\}$ because $\delta h_\xi-\delta p$ obviously belongs to $F_{\xi+1}$. Let $i\in\delta h_\xi-\delta p$. Then there is a $\langle q, L\rangle\in H$ such that $i\in\delta q$ as well as an $\langle r, M\rangle\in H$ extending both $\langle p, K\rangle$ and $\langle q, L\rangle$. Now $i\in\delta r-\delta p$ and hence $r(i)$ must satisfy $K$ in $\mathfrak{B}_\xi(i)$. But $r(i)=h_\xi(i)=\bar{h}_\xi(i)$ and $\varphi\in K$ so that $\bar{h}_\xi(i)$ satisfies $\varphi$ in $\mathfrak{B}_\xi(i)$. Thus we can now conclude that a sequence $\{F_\xi|\xi<\aleph\}$ of filters satisfying (i)–(vi) can be defined. Let $F=\bigcup\{F_\xi|\xi<\aleph\}$. We claim that $F$ satisfies the conclusion of our theorem. First, it is clear from the construction that $F$ is a proper nonprincipal ultrafilter. Now consider any sequence $\{\mathfrak{A}_i|i<\omega\}$ as in the hypothesis of our theorem and let $\mathfrak{B}=\langle\prod_{i<\omega}\mathfrak{A}_i, b_\xi\rangle_{\xi<\aleph}$ and $\Delta\in S_\aleph(L(\mathfrak{B}))$. Suppose that $SH_\mathfrak{B}(F,\Delta)$. This says that a certain collection of fewer than $\aleph$ subsets of $\omega$ is a subset of $F$. Then by the regularity of $\aleph$ we can find an $\alpha<\aleph$ such that $SH_\mathfrak{B}(F_\xi,\Delta)$ for every $\xi>\alpha$. Choose $\xi>\alpha$ such that $\langle\mathfrak{B}_\xi, \Delta_\xi\rangle=\langle\mathfrak{B},\Delta\rangle$. But then we have $SC_{\mathfrak{B}_\xi}(F_{\xi+1},\Delta_\xi)$ and hence $SC_\mathfrak{B}(F,\Delta)$. Q.E.D.

4. **Some additions.** By a theorem of Tarski we know that there are $\beth$ ultrafilters on $\omega$, where $\beth=\text{Card } S(\aleph)$. How many of them satisfy Theorem 2?

THEOREM 3. *If* MA *then there are* $\beth$ *nonprincipal ultrafilters $F$ on $\omega$ such that for every sequence $\{\mathfrak{A}_i|i<\omega\}$ of countable structures of the same type and length $<\aleph$ the ultraproduct $\prod_{i<\omega}\mathfrak{A}_i/F$ is saturated.*

In order to prove Theorem 3 we need

LEMMA 4. *If* MA *then for any $F\subseteq S^*(\omega)$ with* $\text{Card}(F)<\aleph$ *there exists an $X\in S^*(\omega)$ such that if $F$ is a proper filter then so is $((F\cup\{X\}))$ and $((F\cup\{\omega-X\}))$.*

PROOF. Let $P=H_\omega(\omega,\{0,1\})$ and for $p, q\in P$ put $p\leqq q$ if $p\subseteq q$. Clearly $\mathscr{P}=\langle P,\leqq\rangle$ is a partially ordered set that satisfies the c.a.c. For each $Y\in F$ define $D_Y=\{p\in P|(\exists i,j\in\delta p\cap Y)\ p(i)=0$ and $p(j)=1\}$. Since each $Y\in F$ is infinite $D_Y$ is dense. Hence $Z=\{D_Y|Y\in F\}$ is a collection of less than $\aleph$ dense subsets of $P$. Now use MA to get a $Z$-generic $H\subseteq P$ and let $h=\bigcup H$. Then clearly $X=\{i\in\omega|h(i)=1\}$ satisfies the lemma.

Q.E.D.

PROOF OF THEOREM 3. Use Lemma 4 to get a map $r$ from $S_\aleph(S^*(\omega))$ into $S^*(\omega)$ such that $r(F)$ is some $X$ which satisfies the conclusion of Lemma 4 for $F$. Let $A=\{\xi<\aleph|\xi$ is a limit ordinal$\}$. Clearly $\text{Card}(2^A)=\beth$.

We will show that corresponding to each $f \in 2^A$ there is a distinct $F_f$ satisfying Theorem 2. Consider some $f \in 2^A$. Run through the construction in the proof of Theorem 2 as before, only this time replace condition (vi) by the new condition $(\text{vi})_f$: if $\xi$ is a limit ordinal take $F_\xi$ to be $((K_\xi \cup \{r(K_\xi)\}))$ if $f(\xi)=1$ and to be $((K_\xi \cup \{\omega - r(K_\xi)\}))$ otherwise, where $K_\xi = \bigcup \{F_\alpha | \alpha < \xi\}$. Let $F_f$ be the resulting ultrafilter. It is easy to verify that if $f, g \in 2^A$ and $f \neq g$ then $F_f$ and $F_g$ are distinct ultrafilters satisfying Theorem 2. Q.E.D.

An interesting consequence of Lemma 4 is

COROLLARY 5. *If* MA *then no nonprincipal proper ultrafilter on* $\omega$ *is generated by fewer than* $\aleph$ *of its members.*

That Corollary 5 need not hold in the absence of MA is pointed out in [**2**].

Can Theorem 1 be extended, in analogy with (2′), to all $\mathfrak{A}$ of length $<\aleph$ and cardinality $\leqq \aleph$?

THEOREM 6. *If* $\aleph > \aleph_1$ *then for any nonprincipal ultrafilter* $F$ *on* $\omega$ *the ultrapower* $\mathfrak{B} = \langle \aleph_1, \leqq \rangle^\omega / F$ *is not saturated.*

PROOF. The cardinality of $\mathfrak{B}$ is $\geqq \aleph > \aleph_1$ so that it will be enough to show that $\mathfrak{B}$ is not $\aleph_2$-saturated. For each $\xi \in \aleph_1$ let $b_\xi$ be the function with domain $\omega$ and range $\{\xi\}$. Let $\mathfrak{B}' = \langle \mathfrak{B}, b_\xi \rangle_{\xi < \aleph_1}$ and let $c_\xi$ be a constant in $L(\mathfrak{B}')$ denoting $b_\xi$. For $\Delta$ take $\{c_\xi \leqq v_0 | \xi \in \aleph_1\}$ and notice that $\mathrm{SH}_{\mathfrak{B}'}(F, \Delta)$ is true. We claim that $\mathrm{SC}_{\mathfrak{B}'}(F, \Delta)$ is false. For otherwise let $f \in \aleph_1^\omega$ satisfy $\Delta$ in $\mathfrak{B}'$. Define $S_\xi = \{i \in \omega | b_\xi(i) \leqq f(i)\}$ and notice that $S_\eta \subseteq S_\xi$ for $\xi < \eta < \aleph_1$ and $S_\xi \in F$ for every $\xi < \aleph_1$. Thus there is a $\xi_0$ such that $S_{\xi_0} = S_\xi$ for every $\xi_0 < \xi$. But then if $i \in S_{\xi_0}$ we must have $i \in \bigcap \{S_\xi | \xi < \aleph_1\}$ which implies that $\aleph_1 \leqq f(i)$, a contradiction. Q.E.D.

Thus the limitation on the cardinality of $\mathfrak{A}$ in Theorem 1 cannot be removed. Finally let us return to the claim made just after (2). Let $\lambda$ be the least cardinal such that $\aleph < \text{Card } S(\lambda)$. If MA then by Theorem 1 and (4) there will be a nonprincipal ultrafilter $F$ on $\omega$ such that if $\mathfrak{A}$ is a countably infinite structure of countable length then $\mathfrak{A}^\omega / F$ is a $\lambda$-saturated elementary extension of $\mathfrak{A}$ having cardinality $\aleph$. Now the index $\omega$ of this ultraproduct is very small. In the next theorem we show (in the absence of MA) that it is consistent that for no $\kappa$ with $\kappa^+ < \lambda$ and for no nonprincipal ultrafilter $F$ on $\kappa$ is $\mathfrak{A}^\kappa / F$ $\lambda$-saturated, i.e. that the index of the ultraproduct must be as large as possible in order to get $\lambda$-saturation.

THEOREM 7. *There is a Boolean valued model* $V^{\mathscr{B}}$ *of set theory in which* $\lambda = \aleph = \aleph_2$ *and* $\mathfrak{B} = \langle \omega, \leqq \rangle^\omega / F$ *is* $\lambda$*-saturated for no nonprincipal ultrafilter* $F$ *on* $\omega$.

PROOF. Most of our argument is reconstructed from [5]. Without loss of generality we may assume that $V=L$. Let $X=\{0,1\}^{\aleph_2}$ with a topology generated by the subbasic open sets $B_{\xi n}=\{f\in X \mid f(\xi)=n\}$ and let $\mu$ be a product measure on the Borel sets of $X$ which is induced by the unbiased measure on $\{0,1\}$. Our complete Boolean algebra $\mathscr{B}$ is the quotient of the Borel sets of $X$ mod those Borel sets of measure 0. Cardinals are preserved when extending $V$ to $V^{\mathscr{B}}$ because $\mathscr{B}$ satisfies the c.a.c. and $V^{\mathscr{B}}$ satisfies $\lambda=\aleph=\aleph_2$ by the usual argument. Since $\mathscr{B}$ is a measure algebra it satisfies the $(\omega,\omega)$-weak distributive law

$$\prod_{n\in\omega}\sum_{m\in\omega} b_{nm}=\sum_{s\in\omega^\omega}\prod_{n\in\omega}\sum_{m\leqq s(n)} b_{nm}$$

where $b:\omega\times\omega\to\mathscr{B}$, and consequently

$$\begin{aligned}\|f:\check\omega\to\check\omega\| &\leqq \prod_{n\in\omega}\sum_{m\in\omega}\|f(\check n)=\check m\|\\ &=\sum_{s\in\omega\omega}\prod_{n\in\omega}\sum_{m\leqq s(n)}\|f(\check n)=m\|\\ &=\|(\exists s\in\check\omega^{\omega})\,(\forall n\in\check\omega)f(n)\leqq s(n)\|.\end{aligned}$$

Details can be found in [5]. Thus in $V^{\mathscr{B}}$ there is a set $G$ of $\aleph_1$ functions $g\in\omega^\omega\cap V$ which dominate every one of the $\aleph_2$ functions $f\in\omega^\omega$ in the sense that $f<g$ if $(\forall n\in\omega)f(n)\leqq g(n)$. Then working in $V^{\mathscr{B}}$, let $\{b_\xi \mid \xi<\aleph_1\}$ enumerate $G$, let $\mathfrak{B}'=\langle\mathfrak{B},b_\xi\rangle_{\xi<\aleph_1}$, and let $c_\xi$ be a constant in $L(\mathfrak{B}')$ denoting $b_\xi$. For $\Delta$ take $\{c_\xi\leqq v_0 \mid \xi\in\aleph_1\}$ and notice that $\mathrm{SH}_{\mathfrak{B}'}(F,\Delta)$ is true. $\mathrm{SC}_{\mathfrak{B}'}(F,\Delta)$ is false because otherwise there would be a function $f$ which is dominated by no $g\in G$. Q.E.D.

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