

A 4-COLOR THEOREM FOR TOROIDAL GRAPHS

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ABSTRACT. It is well known that any graph imbedded in the torus has chromatic number at most seven, and that seven is attained by the graph K_7 . In this note we show that any toroidal graph containing no triangles has chromatic number at most four, and produce an example attaining this upper bound. The results are then extended for arbitrary girth.

1. Introduction. The *chromatic number* of a graph G , denoted by $\chi(G)$, is the minimum number of colors that can be assigned to the points of G so that adjacent points are colored differently. The Four Color Conjecture is that any planar graph has chromatic number at most four; the Five Color Theorem states that $\chi(G) \leq 5$, if G is planar. Grötzsch [2] has shown that a planar graph having no triangles has chromatic number at most three; the cycle C_5 shows that this bound cannot be improved. It is well known that any toroidal graph has chromatic number at most seven, and that this bound is attained by the complete graph K_7 . In this note we find an upper bound for the chromatic number of toroidal graphs having no triangles, and show that this bound is best possible. We also consider toroidal graphs of arbitrary girth.

A connected graph G is said to be *n-line-critical* if $\chi(G) = n$ but, for any line e of G , $\chi(G - e) = n - 1$. We will use the following theorem of Dirac [1]; p and q denote the number of points and lines of G , respectively.

THEOREM 1. *If G is an n -line-critical graph, $n \geq 4$, and if $G \neq K_n$, then $2q \geq (n-1)p + n - 3$.*

We will also use the following well-known consequence of the generalized euler polyhedral formula (see, for example, [3, p. 118]):

THEOREM 2. *If G is a connected graph of genus γ and having no triangles, then $\gamma \geq 1 - p/2 + q/4$.*

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2. **The main result.** We are now prepared to extend Grötzsch's theorem to the torus.

THEOREM 3. *If a graph G having no triangles is imbedded in the torus, then $\chi(G) \leq 4$.*

PROOF. Let $\chi(G) = n \geq 5$. We first assume that G is n -line-critical, and hence connected. Since G has no triangles, $G \neq K_n$. By Theorem 1,

$$(1) \quad 2q \geq (n-1)p + n - 3.$$

Now if G has genus one, then by Theorem 2,

$$(2) \quad 4p \geq 2q.$$

Combining (1) and (2), we see that $n \leq 4$. If G has genus zero, then $n \leq 3$, by Grötzsch's theorem. In either case we have a contradiction, so that $n \leq 4$.

Now suppose that G is not n -line-critical. Then G contains an n -line-critical subgraph H , and the argument above shows that $\chi(G) = \chi(H) = n \leq 4$.

The graph of Figure 1, constructed by Mycielski [4] as an example of a graph having no triangles and chromatic number four, also has genus one, so that the bound of Theorem 3 cannot be improved.

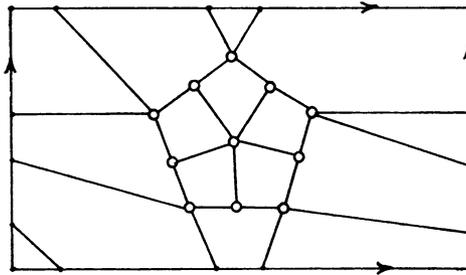


FIGURE 1

3. **Extensions for arbitrary girth.** The *girth* $g(G)$ of a graph G is the length of a shortest cycle (if any) in G . A graph G with cycles but no triangles has $g(G) \geq 4$; the graph of Figure 1 has girth exactly four.

If $g(G) \geq 6$, then each region in an imbedding for G has at least six edges in its boundary, so that $2q \geq 6r$, where r is the total number of regions. As before, we may assume that $\gamma(G) = 1$ and that G is n -line-critical. If $n \geq 4$, then $2q \geq 3p + 1$, by Theorem 1. By the Euler formula for the torus,

then:

$$0 = p - q + r \leq \frac{2q - 1}{3} - q + \frac{q}{3} = -\frac{1}{3},$$

an obvious contradiction. Hence toroidal graphs of girth six or more have chromatic number three or less. This bound is also best possible, as an appropriate subdivision of the Petersen graph (shown imbedded in the torus in Figure 2) can always be found, having girth m ($m \geq 5$), genus one, and chromatic number three.

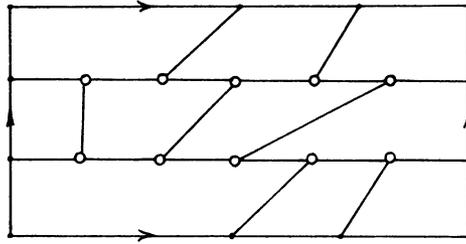


FIGURE 2

For $g(G)=5$ and $\gamma(G)=1$, it follows from Theorem 3 that $\chi(G) \leq 4$. This bound can also be established by employing the following result of Szekeres and Wilf [5]:

THEOREM 4. For any graph G , $\chi(G) \leq 1 + \max \delta(G')$, where the maximum is taken over all induced subgraphs G' of G , and $\delta(G')$ denotes the minimum degree of G' .

For toroidal graphs G' having girth 5, if $\delta(G') \geq 4$, then $2q \geq 4p$ and also $2q \geq 5r$, so that

$$0 = p - q + r \leq \frac{q}{2} - q + \frac{2}{5}q = -\frac{q}{10};$$

hence $\delta(G') \leq 3$. It now follows from Theorem 4 that $\chi(G) \leq 4$. Whether this upper bound can be attained remains an open question.

We summarize our results in:

THEOREM 5. Let G be imbedded in the torus, with $g(G)=m$. Then

$$\begin{aligned} \chi(G) &\leq 7, & \text{if } m = 3, \\ &\leq 4, & \text{if } m = 4 \text{ or } 5, \\ &\leq 3, & \text{if } m \geq 6. \end{aligned}$$

Furthermore, all the bounds are sharp, except possibly for $m=5$.

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