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NONCONSTANT ENDOMORPHISMS OF LATTICES

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ABSTRACT. There is a proper class of pairwise nonisomorphic lattices whose monoids of all nonconstant endomorphisms are isomorphic to a given monoid M.

1. **Introduction.** There are monoids not appearing as full endomorphism monoids of lattices: every constant mapping of a lattice into itself is one of its endomorphisms and therefore any monoid of all endomorphisms of a lattice contains a left zero element. The nonconstant endomorphisms do not have to form a monoid since the composition of two nonconstant mappings may be a constant mapping.

The aim of the present note is to show that for every monoid M there is a lattice L such that the set of all its nonconstant endomorphisms is closed under composition and isomorphic to M; this solves completely Problem 3 of [**2**]. R. McKenzie advised the author that the result follows from [**3**] under the generalized continuum hypothesis. However, neither the present paper nor any of the results used here requires any set-theoretical assumptions.

Using graph-theoretic and lattice-theoretic results we will prove a substantially stronger theorem which will also yield the number of nonisomorphic lattices with isomorphic monoids of nonconstant endomorphisms.

2. **Graphs.** By a graph we will always mean a pair $\langle X, R\rangle$ in which X is a set and R is a set of two-element subsets of X; a compatible mapping $f:\langle X, R\rangle\to\langle X', R'\rangle$ is a mapping $f:X\to X'$ for which $\{x_1, x_2\}\in R$ implies $\{f(x_1), f(x_2)\}\in R'$. Let $\boldsymbol{G}$ be the category of all graphs and all their compatible mappings and let $\boldsymbol{H}$ be the full subcategory of $\boldsymbol{G}$ determined by all Hell graphs, i.e. by the graphs $\langle X, R\rangle$ satisfying the following condition:

(H) For every x in X there is a three-element set $\{x, x_1, x_2\}\subset X$ such that $\{x, x_1\}$, $\{x, x_2\}$, $\{x_1, x_2\}\in R$.

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In other words, every vertex belongs to a triangle of the graph $\langle X, R\rangle$.

The following theorem is due to P. Hell [**6**].

THEOREM 1. *Every full category of algebras is isomorphic to a full subcategory of* $\boldsymbol{H}$, *i.e.* $\boldsymbol{H}$ *is binding.*

As every monoid $\boldsymbol{M}$ is isomorphic to the full endomorphism monoid of an algebra, the above theorem yields the existence of a Hell graph H with the monoid of all compatible mappings $f: H\to H$ isomorphic to $\boldsymbol{M}$.

3. **Lattices.** Let $\boldsymbol{L}$ be the category of all lattices and all their homomorphisms and let $\boldsymbol{N}$ be the class of all lattices and all nonconstant homomorphisms between them; note that $\boldsymbol{N}$ is not a category.

THEOREM 2. *There is a full subcategory* $\boldsymbol{M}$ *of the category* $\boldsymbol{L}$ *of all lattices such that*

(i) $\boldsymbol{M}\cap\boldsymbol{N}$ *is a category,*

(ii) $\boldsymbol{M}\cap\boldsymbol{N}$ *is isomorphic to* $\boldsymbol{H}$.

In other words, $\boldsymbol{M}\cap\boldsymbol{N}$ is a binding category. To prove the theorem a one-to-one and full functor $M:\boldsymbol{H}\to\boldsymbol{N}$ will be constructed; the objects of $\boldsymbol{M}$ will be all lattices $M(G)$ for $G=\langle X, R\rangle$ in $\boldsymbol{H}$. To define $M(G)$, consider the lattice $F(X)$ freely generated by X and define Θ_R to be the smallest congruence relation on $F(X)$ identifying $x\wedge y$ with $z\wedge t$ and $x\vee y$ with $z\vee t$ whenever both $\{x, y\}$ and $\{z, t\}$ belong to R. Set $M(G)=F(X)/\Theta_R$. Let $\pi_G: F(X)\to M(G)$ be the canonical homomorphism with $\operatorname{Ker}\pi_G=\Theta_R$, let $f: G\to G'=\langle X', R'\rangle$ be a compatible mapping and let $F(f): F(X)\to F(X')$ be the homomorphism of the free lattices extending f. If $\{x_1, x_2\}\in R$, then there is x_3 in X, $x_3\neq x_2$, such that $\{x_1, x_3\}\in R$. Hence $x_1\vee x_2\equiv x_1\vee x_3(\Theta_R)$ and $x_1\wedge x_2\equiv x_1\wedge x_3(\Theta_R)$. As f is a compatible mapping, then $F(f)(x_1\vee x_2)=F(f)(x_1)\vee F(f)(x_2)=f(x_1)\vee f(x_2)\equiv f(x_1)\vee f(x_3)=F(f)(x_1\vee x_3)$ in $\Theta_{R'}$. Similarly, $F(f)(x_1\wedge x_2)\equiv F(f)(x_1\wedge x_3)(\Theta_{R'})$. Consequently, $\Theta_R\subseteq\operatorname{Ker}(\pi_{G'}\circ F(f))$ so there is a unique homomorphism

$$M(f): M(G)\to M(G')$$

such that $M(f)\circ\pi_G=\pi_{G'}\circ F(f)$. It is easy to see that M is a functor from $\boldsymbol{H}$ into the category $\boldsymbol{L}$ of all lattices. Denote $e=\pi_G(x\vee y)=\pi_G(\bar{x}\vee\bar{y})$ for any $\{x, y\}$, $\{\bar{x}, \bar{y}\}$ in R; similarly, $z=\pi_G(x\wedge y)$. Since the graphs satisfying (H) do not have isolated points, $z\leqq\pi_G(x)\leqq e$ for every x in X; as $M(G)$ is generated by $\pi_G(X)$, $z\leqq m\leqq e$ for every m in $M(G)$. Note also, that for $\{x_1, x_2\}$ in R, $\{\pi_G(x_1), \pi_G(x_2)\}$ is a complemented pair of elements of $M(G)$, so that, for every compatible $f: G\to G'$, $M(f)(e)=e'$ and $M(f)(z)=z'$.

The following lemma is an easy consequence of Theorem 2 of [**3**].

LEMMA 1. (1) *If* $m \in M(G)\backslash\{e, z\}$, *then* $|\pi_G^{-1}(m)|=1$.

(2) $\{m_1, m_2\}$ *is a complemented pair of elements of* $M(G)$ *if and only if either* $\{m_1, m_2\}=\{e, z\}$ *or* $\{m_1, m_2\}=\{\pi_G(x_1), \pi_G(x_2)\}$ *for some* $\{x_1, x_2\}$ *in* R.

LEMMA 2. (3) *The functor* M *is one-to-one*, $M:\boldsymbol{H}\to\boldsymbol{N}$.

(4) If $\varphi:M(G)\to M(G')$ *is a lattice homomorphism such that* $\varphi(e)=e'$ *and* $\varphi(z)=z'$, *then* $\varphi=M(f)$ *for a compatible mapping* $f:G\to G'$.

PROOF. (3) follows immediately from (1) and (H).

(4) φ preserves all complemented pairs. Since every vertex x of $\langle X, R\rangle$ belongs to a triangle of R, $\varphi(x) \in M(G')\backslash\{e', z'\}$ by (2). The only other elements of $M(G')$ possessing complements are elements of the form $\pi_{G'}(y)$ for y in X' so that $\varphi(\pi_G(X))\subseteq \pi_{G'}(X')$. Using (1) and (2) again, we conclude that there is a compatible mapping $f:G\to G'$ with $M(f)\circ\pi_G=\pi_{G'}\circ F(f)=\varphi\circ\pi_G$. Since π_G is an onto homomorphism, $\varphi=M(f)$.

To finish the proof of the theorem it remains to show that all the other homomorphisms $\varphi:M(G)\to M(G')$ are constant.

If $\varphi(z)=e'$ or $\varphi(e)=z'$, then φ is constant, as $z\leqq m\leqq e$ for all elements m of $M(G)$, and therefore we may assume that $d=\varphi(z)\neq e', z'$. For any triangle $\{a, b, c\}$ of G, $A=\{\pi_G(a), \pi_G(b), \pi_G(c), e, z\}$ is a simple sublattice of $M(G)$; set $\pi_G(s)=\bar{s}$ for every s in $F(X)$. Consider the sublattice $\varphi(A)$ of $M(G')$. If φ is not constant, then $\varphi|A$ has to be a one-to-one homomorphism. If $\varphi(\bar{a})=z'$, then $\varphi(z)\leqq\varphi(\bar{a})=z'$, a contradiction. $\varphi(\bar{a})=e'$ implies $\varphi(\bar{b})=e'\wedge\varphi(\bar{b})=\varphi(\bar{a}\wedge\bar{b})=\varphi(z)$, a contradiction again. We may assume that $\varphi(\bar{a}), \varphi(\bar{b}), \varphi(\bar{c}) \notin \{e', z'\}$; since $d=\varphi(\bar{a})\wedge\varphi(\bar{b})=\varphi(\bar{a})\wedge\varphi(\bar{c})$, using (1) we obtain the equation

$$\pi_{G'}^{-1}(d) = \pi_{G'}^{-1}(\varphi(\bar{a})) \wedge \pi_{G'}^{-1}(\varphi(\bar{b})) = \pi_{G'}^{-1}(\varphi(\bar{a})) \wedge \pi_{G'}^{-1}(\varphi(\bar{c}))$$

in the free lattice $F(X')$. By Lemma 2.6 of [**6**],

$$\pi_{G'}^{-1}(d) = \pi_{G'}^{-1}(\varphi(\bar{a})) \wedge (\pi_{G'}^{-1}(\varphi(\bar{b})) \vee \pi_{G'}^{-1}(\varphi(\bar{c})))$$

in $F(X')$; consequently,

$$d = \varphi(\bar{a}) \wedge (\varphi(\bar{b}) \vee \varphi(\bar{c})) = \varphi(\bar{a} \wedge (\bar{b} \vee \bar{c})) \quad \text{in } M(G').$$

But $\bar{b}\vee\bar{c}=\bar{a}\vee\bar{c}\geqq\bar{a}$, therefore $\varphi(z)=d=\varphi(\bar{a})$; $\varphi|A$ cannot be one-to-one, thus φ is a constant homomorphism of $M(G)$ into $M(G')$.

4. **Number of representations.** Note that $|M(X, R)|=|X|$ for any infinite X. Utilizing this together with the existence of a full embedding of the category of commutative groupoids into the category $\boldsymbol{H}$ by a functor K preserving the infinite cardinalities of underlying sets (the existence of such a functor K follows easily from [**4**] and [**6**]), we obtain the following theorem as an immediate consequence of Theorem 5 of [**5**].

THEOREM 3. *Let M be a monoid, $|M|=m$ and let n be an infinite cardinal number, $n \geqq m$. Then there are exactly 2^n lattices L_α such that*

(a) *the set of all nonconstant endomorphisms of each L_α is isomorphic to M,*

(b) $|L_\alpha|=n$ *for each* $\alpha \in 2^n$,

(c) *given indices* $\alpha, \alpha' \in 2^n$, $\alpha \neq \alpha'$, $\mathrm{Hom}_L(L_\alpha, L_{\alpha'})$ *consists exactly of all constant homomorphisms.*

COROLLARY. *For every infinite cardinal number n there is a lattice L of cardinality n such that its endomorphisms are constant mappings only.*

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