

## NONCONSTANT ENDOMORPHISMS OF LATTICES

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**ABSTRACT.** There is a proper class of pairwise nonisomorphic lattices whose monoids of all nonconstant endomorphisms are isomorphic to a given monoid  $M$ .

**1. Introduction.** There are monoids not appearing as full endomorphism monoids of lattices: every constant mapping of a lattice into itself is one of its endomorphisms and therefore any monoid of all endomorphisms of a lattice contains a left zero element. The nonconstant endomorphisms do not have to form a monoid since the composition of two nonconstant mappings may be a constant mapping.

The aim of the present note is to show that for every monoid  $M$  there is a lattice  $L$  such that the set of all its nonconstant endomorphisms is closed under composition and isomorphic to  $M$ ; this solves completely Problem 3 of [2]. R. McKenzie advised the author that the result follows from [3] under the generalized continuum hypothesis. However, neither the present paper nor any of the results used here requires any set-theoretical assumptions.

Using graph-theoretic and lattice-theoretic results we will prove a substantially stronger theorem which will also yield the number of nonisomorphic lattices with isomorphic monoids of nonconstant endomorphisms.

**2. Graphs.** By a graph we will always mean a pair  $\langle X, R \rangle$  in which  $X$  is a set and  $R$  is a set of two-element subsets of  $X$ ; a compatible mapping  $f: \langle X, R \rangle \rightarrow \langle X', R' \rangle$  is a mapping  $f: X \rightarrow X'$  for which  $\{x_1, x_2\} \in R$  implies  $\{f(x_1), f(x_2)\} \in R'$ . Let  $G$  be the category of all graphs and all their compatible mappings and let  $H$  be the full subcategory of  $G$  determined by all Hell graphs, i.e. by the graphs  $\langle X, R \rangle$  satisfying the following condition:

For every  $x$  in  $X$  there is a three-element  
(H) set  $\{x, x_1, x_2\} \subset X$  such that  $\{x, x_1\}$ ,  
 $\{x, x_2\}, \{x_1, x_2\} \in R$ .

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In other words, every vertex belongs to a triangle of the graph  $\langle X, R \rangle$ . The following theorem is due to P. Hell [6].

**THEOREM 1.** *Every full category of algebras is isomorphic to a full subcategory of  $\mathbf{H}$ , i.e.  $\mathbf{H}$  is binding.*

As every monoid  $\mathbf{M}$  is isomorphic to the full endomorphism monoid of an algebra, the above theorem yields the existence of a Hell graph  $H$  with the monoid of all compatible mappings  $f: H \rightarrow H$  isomorphic to  $\mathbf{M}$ .

**3. Lattices.** Let  $\mathbf{L}$  be the category of all lattices and all their homomorphisms and let  $\mathbf{N}$  be the class of all lattices and all nonconstant homomorphisms between them; note that  $\mathbf{N}$  is not a category.

**THEOREM 2.** *There is a full subcategory  $\mathbf{M}$  of the category  $\mathbf{L}$  of all lattices such that*

- (i)  $\mathbf{M} \cap \mathbf{N}$  is a category,
- (ii)  $\mathbf{M} \cap \mathbf{N}$  is isomorphic to  $\mathbf{H}$ .

In other words,  $\mathbf{M} \cap \mathbf{N}$  is a binding category. To prove the theorem a one-to-one and full functor  $M: \mathbf{H} \rightarrow \mathbf{N}$  will be constructed; the objects of  $\mathbf{M}$  will be all lattices  $M(G)$  for  $G = \langle X, R \rangle$  in  $\mathbf{H}$ . To define  $M(G)$ , consider the lattice  $F(X)$  freely generated by  $X$  and define  $\Theta_R$  to be the smallest congruence relation on  $F(X)$  identifying  $x \wedge y$  with  $z \wedge t$  and  $x \vee y$  with  $z \vee t$  whenever both  $\{x, y\}$  and  $\{z, t\}$  belong to  $R$ . Set  $M(G) = F(X) / \Theta_R$ . Let  $\pi_G: F(X) \rightarrow M(G)$  be the canonical homomorphism with  $\text{Ker } \pi_G = \Theta_R$ , let  $f: G \rightarrow G' = \langle X', R' \rangle$  be a compatible mapping and let  $F(f): F(X) \rightarrow F(X')$  be the homomorphism of the free lattices extending  $f$ . If  $\{x_1, x_2\} \in R$ , then there is  $x_3$  in  $X$ ,  $x_3 \neq x_2$ , such that  $\{x_1, x_3\} \in R$ . Hence  $x_1 \vee x_2 \equiv x_1 \vee x_3 (\Theta_R)$  and  $x_1 \wedge x_2 \equiv x_1 \wedge x_3 (\Theta_R)$ . As  $f$  is a compatible mapping, then  $F(f)(x_1 \vee x_2) = F(f)(x_1) \vee F(f)(x_2) = f(x_1) \vee f(x_2) \equiv f(x_1) \vee f(x_3) = F(f)(x_1 \vee x_3)$  in  $\Theta_{R'}$ . Similarly,  $F(f)(x_1 \wedge x_2) \equiv F(f)(x_1 \wedge x_3) (\Theta_{R'})$ . Consequently,  $\Theta_R \subseteq \text{Ker}(\pi_{G'} \circ F(f))$  so there is a unique homomorphism

$$M(f): M(G) \rightarrow M(G')$$

such that  $M(f) \circ \pi_G = \pi_{G'} \circ F(f)$ . It is easy to see that  $M$  is a functor from  $\mathbf{H}$  into the category  $\mathbf{L}$  of all lattices. Denote  $e = \pi_G(x \vee y) = \pi_G(\bar{x} \vee \bar{y})$  for any  $\{x, y\}, \{\bar{x}, \bar{y}\}$  in  $R$ ; similarly,  $z = \pi_G(x \wedge y)$ . Since the graphs satisfying (H) do not have isolated points,  $z \leq \pi_G(x) \leq e$  for every  $x$  in  $X$ ; as  $M(G)$  is generated by  $\pi_G(X)$ ,  $z \leq m \leq e$  for every  $m$  in  $M(G)$ . Note also, that for  $\{x_1, x_2\}$  in  $R$ ,  $\{\pi_G(x_1), \pi_G(x_2)\}$  is a complemented pair of elements of  $M(G)$ , so that, for every compatible  $f: G \rightarrow G'$ ,  $M(f)(e) = e'$  and  $M(f)(z) = z'$ .

The following lemma is an easy consequence of Theorem 2 of [3].

LEMMA 1. (1) If  $m \in M(G) \setminus \{e, z\}$ , then  $|\pi_G^{-1}(m)| = 1$ .

(2)  $\{m_1, m_2\}$  is a complemented pair of elements of  $M(G)$  if and only if either  $\{m_1, m_2\} = \{e, z\}$  or  $\{m_1, m_2\} = \{\pi_G(x_1), \pi_G(x_2)\}$  for some  $\{x_1, x_2\}$  in  $R$ .

LEMMA 2. (3) The functor  $M$  is one-to-one,  $M: \mathbf{H} \rightarrow \mathbf{N}$ .

(4) If  $\varphi: M(G) \rightarrow M(G')$  is a lattice homomorphism such that  $\varphi(e) = e'$  and  $\varphi(z) = z'$ , then  $\varphi = M(f)$  for a compatible mapping  $f: G \rightarrow G'$ .

PROOF. (3) follows immediately from (1) and (H).

(4)  $\varphi$  preserves all complemented pairs. Since every vertex  $x$  of  $\langle X, R \rangle$  belongs to a triangle of  $R$ ,  $\varphi(x) \in M(G') \setminus \{e', z'\}$  by (2). The only other elements of  $M(G')$  possessing complements are elements of the form  $\pi_{G'}(y)$  for  $y$  in  $X'$  so that  $\varphi(\pi_G(X)) \subseteq \pi_{G'}(X')$ . Using (1) and (2) again, we conclude that there is a compatible mapping  $f: G \rightarrow G'$  with  $M(f) \circ \pi_G = \pi_{G'} \circ F(f) = \varphi \circ \pi_G$ . Since  $\pi_G$  is an onto homomorphism,  $\varphi = M(f)$ .

To finish the proof of the theorem it remains to show that all the other homomorphisms  $\varphi: M(G) \rightarrow M(G')$  are constant.

If  $\varphi(z) = e'$  or  $\varphi(e) = z'$ , then  $\varphi$  is constant, as  $z \leq m \leq e$  for all elements  $m$  of  $M(G)$ , and therefore we may assume that  $d = \varphi(z) \neq e', z'$ . For any triangle  $\{a, b, c\}$  of  $G$ ,  $A = \{\pi_G(a), \pi_G(b), \pi_G(c), e, z\}$  is a simple sublattice of  $M(G)$ ; set  $\pi_G(s) = \bar{s}$  for every  $s$  in  $F(X)$ . Consider the sublattice  $\varphi(A)$  of  $M(G')$ . If  $\varphi$  is not constant, then  $\varphi|_A$  has to be a one-to-one homomorphism. If  $\varphi(\bar{a}) = z'$ , then  $\varphi(z) \leq \varphi(\bar{a}) = z'$ , a contradiction.  $\varphi(\bar{a}) = e'$  implies  $\varphi(\bar{b}) = e' \wedge \varphi(\bar{b}) = \varphi(\bar{a} \wedge \bar{b}) = \varphi(z)$ , a contradiction again. We may assume that  $\varphi(\bar{a}), \varphi(\bar{b}), \varphi(\bar{c}) \notin \{e', z'\}$ ; since  $d = \varphi(\bar{a}) \wedge \varphi(\bar{b}) = \varphi(\bar{a}) \wedge \varphi(\bar{c})$ , using (1) we obtain the equation

$$\pi_{G'}^{-1}(d) = \pi_{G'}^{-1}(\varphi(\bar{a})) \wedge \pi_{G'}^{-1}(\varphi(\bar{b})) = \pi_{G'}^{-1}(\varphi(\bar{a})) \wedge \pi_{G'}^{-1}(\varphi(\bar{c}))$$

in the free lattice  $F(X')$ . By Lemma 2.6 of [6],

$$\pi_{G'}^{-1}(d) = \pi_{G'}^{-1}(\varphi(\bar{a})) \wedge (\pi_{G'}^{-1}(\varphi(\bar{b})) \vee \pi_{G'}^{-1}(\varphi(\bar{c})))$$

in  $F(X')$ ; consequently,

$$d = \varphi(\bar{a}) \wedge (\varphi(\bar{b}) \vee \varphi(\bar{c})) = \varphi(\bar{a} \wedge (\bar{b} \vee \bar{c})) \quad \text{in } M(G').$$

But  $\bar{b} \vee \bar{c} = \bar{a} \vee \bar{c} \geq \bar{a}$ , therefore  $\varphi(z) = d = \varphi(\bar{a})$ ;  $\varphi|_A$  cannot be one-to-one, thus  $\varphi$  is a constant homomorphism of  $M(G)$  into  $M(G')$ .

**4. Number of representations.** Note that  $|M(X, R)| = |X|$  for any infinite  $X$ . Utilizing this together with the existence of a full embedding of the category of commutative groupoids into the category  $\mathbf{H}$  by a functor  $K$  preserving the infinite cardinalities of underlying sets (the existence of such a functor  $K$  follows easily from [4] and [6]), we obtain the following theorem as an immediate consequence of Theorem 5 of [5].

**THEOREM 3.** *Let  $M$  be a monoid,  $|M|=m$  and let  $n$  be an infinite cardinal number,  $n \geq m$ . Then there are exactly  $2^n$  lattices  $L_\alpha$  such that*

- (a) *the set of all nonconstant endomorphisms of each  $L_\alpha$  is isomorphic to  $M$ ,*
- (b)  *$|L_\alpha|=n$  for each  $\alpha \in 2^n$ ,*
- (c) *given indices  $\alpha, \alpha' \in 2^n$ ,  $\alpha \neq \alpha'$ ,  $\text{Hom}_L(L_\alpha, L_{\alpha'})$  consists exactly of all constant homomorphisms.*

**COROLLARY.** *For every infinite cardinal number  $n$  there is a lattice  $L$  of cardinality  $n$  such that its endomorphisms are constant mappings only.*

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