

MEET-IRREDUCIBLE ELEMENTS IN IMPLICATIVE LATTICES¹

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ABSTRACT. A characterization of meet-irreducible elements and atoms in an implicative lattice is obtained and used to derive the following theorems. A complete lattice is implicative and every element has a meet-irreducible decomposition if and only if there are enough principal prime relative annihilator ideals to separate distinct elements. The MacNeille completion of an implicative lattice is an implicative lattice; furthermore the embedding preserves relative pseudocomplements, meet-irreducible elements and atoms.

1. Introduction. An *implicative lattice* is a system $\mathcal{Q}=(L, \leq, \vee, \wedge, \rightarrow, 1)$ such that (L, \vee, \wedge) is a lattice and \rightarrow is a binary operation on L which, for every $a, b, c \in L$, satisfies

$$c \leq a \rightarrow b \quad \text{if and only if} \quad c \wedge a \leq b.$$

The element $a \rightarrow b$ is called the *pseudocomplement of a relative to b* . The variety of implicative lattices and its relationship to logic and other mathematical systems has been extensively studied [2], [8], [9], [11], [12] with considerable lack of uniformity in terminology and notation. The following is a brief list of properties for later reference.

If \mathcal{Q} is an implicative lattice and $x, y, z \in L$, then

- (1.1) $y \leq x \rightarrow y$;
- (1.2) $x \leq y$ if and only if $x \rightarrow y = 1$;
- (1.3) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$;
- (1.4) $(y \wedge z) \rightarrow x = z \rightarrow (y \rightarrow x) = y \rightarrow (z \rightarrow x) = (z \rightarrow y) \rightarrow (z \rightarrow x)$;
- (1.5) L is a distributive lattice;
- (1.6) if $B \subseteq L$ such that $\bigwedge B$ exists then, for every $a \in L$, $\bigwedge \{a \rightarrow y \mid y \in B\}$ exists and equals $a \rightarrow \bigwedge B$ [11, p. 136].

2. Meet-irreducible elements and atoms. In 1938, Ward [13] obtained a necessary condition for an element to be meet-irreducible in a residuated

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lattice. The following theorem shows this condition to be both necessary and sufficient for an element to be meet-irreducible in an implicative lattice.

THEOREM 2.1. *An element m is meet-irreducible in an implicative lattice \mathcal{L} if and only if $x \not\leq m$ implies $x \rightarrow m = m$.*

PROOF OF NECESSITY. Let m be a meet-irreducible element of an implicative lattice \mathcal{L} and $x \in L$ such that $x \not\leq m$. By definition of $x \rightarrow m$ it will suffice to show that $x \wedge z \leq m$ implies $z \leq m$, since clearly $z \leq m$ implies $x \wedge z \leq m$. If $x \wedge z \leq m$ then $m = m \vee (x \wedge z) = (m \vee x) \wedge (m \vee z)$, so $m = m \vee x$ or $m = m \vee z$ since m is meet-irreducible, but $x \not\leq m$ implies $m \neq m \vee x$ thus $m = m \vee z$, i.e. $z \leq m$.

PROOF OF SUFFICIENCY. Let $m \in L$ such that, for every $x \not\leq m$, $x \rightarrow m = m$ and $p, q \in L$ such that $m < p$ and $m = p \wedge q$. Then $p \not\leq m$ so $p \rightarrow m = m = p \wedge q$, so by definition of $p \rightarrow m$ $q \leq m$, but also $m = p \wedge q \leq q$, hence $m = q$; therefore m is meet-irreducible. \square

COROLLARY 2.1.a. *An element m is meet-irreducible in a Boolean algebra if and only if $x \not\leq m$ implies $m' \leq x$.*

COROLLARY 2.1.b. *If b is meet-irreducible then, for every a , $a \rightarrow b$ is meet-irreducible.*

PROOF. If $x \not\leq a \rightarrow b$ then $x \not\leq b \leq a \rightarrow b$ so $x \rightarrow b = b$; hence $x \rightarrow (a \rightarrow b) = a \rightarrow (x \rightarrow b) = a \rightarrow b$, by (1.4). \square

COROLLARY 2.1.c. *If $\tau: L_1 \rightarrow L_2$ is an implicative homomorphism and m is a meet-irreducible element of L_1 then $m\tau$ is a meet-irreducible element of L_2 .*

PROOF. If $x\tau \in L_2$ such that $x\tau \not\leq m\tau$ then $x \not\leq m$ hence $x \rightarrow m = m$ and $(x \rightarrow m)\tau = x\tau \rightarrow m\tau = m\tau$. \square

COROLLARY 2.1.d. *If M is a set of meet-irreducible elements and $b = \bigwedge M$ then, for every a , $a \rightarrow b = \bigwedge \{a \rightarrow m \mid m \in M\} = \bigwedge \{m \mid m \in M \text{ and } a \not\leq m\}$.*

Let $\mathcal{L} = (L, \vee, \wedge, \rightarrow, 1, 0)$ be a pseudo-Boolean algebra, i.e. an implicative lattice with least element, 0, and $x^* = x \rightarrow 0$ denote the pseudocomplement of x .

THEOREM 2.2. *An element a is an atom in a pseudo-Boolean algebra if and only if $a \neq 0$ and $a \not\leq y$ implies $a \rightarrow y = a^*$.*

PROOF OF NECESSITY. Let a be an atom and $y \in L$ such that $a \not\leq y$. It will suffice to show that $z \leq a^*$ if and only if $z \wedge a \leq y$. If $z \leq a^*$ then $z \wedge a = 0$; so, for every y , $z \wedge a \leq y$. Conversely, if $z \wedge a \leq y$ then since a is an atom $z \wedge a = 0$ or $z \wedge a = a$, but $a = z \wedge a \leq y$ contradicts $a \not\leq y$; thus $0 = z \wedge a$, i.e. $z \leq a^*$.

PROOF OF SUFFICIENCY. Let $a \neq 0$, for every y , $a \not\leq y$ implies $a \rightarrow y = a^*$, and $z < a$. Then $z = z \wedge a$ and $a \not\leq z$ so $z \leq a \rightarrow z = a^*$; hence $z = z \wedge a = 0$, i.e. a is an atom. \square

COROLLARY 2.2.a. *If a is an atom then a^* is meet-irreducible.*

PROOF. First note that $x \not\leq a^*$ if and only if $a \not\leq x^*$; so if a is an atom and $x \not\leq a^*$ then $a \rightarrow x^* = a^*$, so by (1.4), $x \rightarrow a^* = x \rightarrow (a \rightarrow x^*) = a \rightarrow (x \rightarrow x^*) = a \rightarrow x^* = a^*$; thus a^* is meet-irreducible by Theorem 2.1. \square

3. Meet-irreducible decompositions and relative annihilator ideals. Balachandran [1, p. 550] has shown that a complete lattice is implicative if every element has a finite meet-prime decomposition and Newman [10, p. 31] that a complete distributive lattice is meet-continuous if every element has a finite meet-irreducible decomposition. The next theorem generalizes these results.

THEOREM 3.1. *If every element of a complete distributive lattice has a meet-irreducible decomposition, then the lattice is implicative.*

PROOF. Let \mathcal{Q} be a complete distributive lattice in which every element has a meet-irreducible decomposition, M be the set of meet-irreducible elements and, for each $x \in L$, $M(x) = \{m \in M \mid x \leq m\}$. Since \mathcal{Q} is complete, a well-defined binary operation is obtained by defining, for every $a, b \in L$, $a \rightarrow b = \bigwedge [M(b) - M(a)]$. Observe that, for every $x \in L$, $1 \in M(x)$, so $a \rightarrow b = 1$ if and only if $M(b) - M(a) = \emptyset$ if and only if $a \leq b$.

McKinsey and Tarski [8, Theorem 1.4] have shown that \mathcal{Q} is implicative provided, for every $a, b, c \in L$, (1) $a \wedge a \rightarrow b \leq b$, (2) $a \leq b \rightarrow (a \wedge b)$, and (3) $c \rightarrow (a \wedge b) \leq c \rightarrow a$. First note that $a \wedge a \rightarrow b \leq b$ is implied by $M(b) \subseteq M(a) \cup [M(b) - M(a)]$, so (1) is established. To establish (2), we use the distributivity to show that $M(a \wedge b) = M(a) \cup M(b)$; for if $q \in M(a \wedge b)$ then $q = (a \wedge b) \vee q = (a \vee q) \wedge (b \vee q)$, so $q = a \vee q$ or $q = b \vee q$, i.e. $q \in M(a) \cup M(b)$; from this it follows that $M(a \wedge b) - M(b) \subseteq M(a)$, so $a = \bigwedge M(a) \leq \bigwedge [M(a \wedge b) - M(b)] = b \rightarrow (a \wedge b)$. Finally, $M(a) \subseteq M(a \wedge b)$, so $M(a) - M(c) \subseteq M(a \wedge b) - M(c)$ and $c \rightarrow (a \wedge b) = \bigwedge [M(c) - M(a \wedge b)] \leq \bigwedge [M(a) - M(c)] = c \rightarrow a$. \square

Mandelker [7] introduced the notion of the *annihilator of a relative to b* $\langle a, b \rangle = \{z \mid z \wedge a \leq b\}$ and showed that a lattice is distributive if and only if every relative annihilator is an ideal. Clearly, a lattice is implicative if and only if every relative annihilator is a principal ideal, in which case $\langle a, b \rangle = (a \rightarrow b)$. Note further that in a distributive lattice there are enough relative annihilator ideals to separate distinct elements, since $a \not\leq b$ implies $b \in \langle a, b \rangle$ but $a \notin \langle a, b \rangle$. Also the relative annihilator $\langle a, b \rangle$ is proper if and only if $a \not\leq b$, so the class of relative annihilators needed to separate elements need contain none for which $a \leq b$.

THEOREM 3.2. *Let \mathcal{Q} be a complete lattice. Then \mathcal{Q} is an implicative lattice in which every element has a meet-irreducible decomposition if and only if there are enough principal prime relative annihilator ideals to separate distinct elements.*

PROOF. If there are enough principal prime relative annihilator ideals to separate elements then there are enough prime ideals to separate elements; hence L is distributive and the principal ideal $(p]$ is prime if and only if p is meet-irreducible. For each $x \in L$, let $M(x) = \{m \in L \mid m \text{ is meet-irreducible and } x \leq m\}$, so $\bigwedge M(x) = \bar{x}$ exists in L and $x \leq \bar{x}$; but if $x < \bar{x}$ then there exists a principal prime ideal $(p]$ such that $x \in (p]$ but $\bar{x} \notin (p]$; hence there exists a meet-irreducible element p such that $p \in M(x)$ but $\bar{x} \not\leq p$ contradicting $\bar{x} = M(x)$; hence $x = \bigwedge M(x)$, and, by Theorem 3.1, \mathcal{Q} is implicative.

Conversely, suppose L is an implicative lattice in which every element has a meet-irreducible decomposition and $b = \bigwedge M(b) \not\leq a = \bigwedge M(a)$. Then, for some $\bar{m} \in M(a)$, $\bar{m} \notin M(b)$, so $b \not\leq \bar{m}$ and $a \leq \bar{m} \leq b \rightarrow \bar{m}$; thus $a \in \langle b, \bar{m} \rangle = (\bar{m}]$ and $b \notin \langle b, \bar{m} \rangle = (\bar{m}]$.

4. The MacNeille completion. MacNeille [6] showed that by generalizing Dedekind's completion by cuts of the rational numbers, any partially ordered set can be embedded in a complete lattice in a manner which preserves the order relation and any existing greatest lower bounds and least upper bounds (i.e. it is a regular embedding). Furthermore, the embedding is both meet and join dense. The embedding does not necessarily preserve other important lattice structure; Funayama [4] constructed a distributive lattice whose MacNeille completion was not even modular, and Crawley [3] constructed a distributive lattice which could not be regularly embedded in any complete modular lattice. The existence of complements or relative pseudocomplements seems however to be helpful in preserving distributivity, as Glivenko [5] and Stone [12] show that the MacNeille completion of a Boolean algebra is a Boolean algebra and we now show that the MacNeille completion of an implicative lattice is an implicative lattice; furthermore the embedding preserves relative pseudocomplements and any existing meet-irreducible elements and atoms.

THEOREM 4.1. *The MacNeille completion of an implicative lattice is an implicative lattice, and the embedding preserves relative pseudocomplements.*

PROOF. Let \mathcal{Q} be an implicative lattice and \mathcal{Q}_e denote its MacNeille completion. Recall [2, Chapter V, §9] that $L_e = \{A^+ \mid A \subseteq L\}$ where A^+ is the set of lower bounds of A in \mathcal{Q} , set inclusion is the partial order and set intersection is the meet operation, and the embedding is $x \rightarrow (x)$, the

principal ideal generated by x . It is easily shown that, for every $B, C \in L_e$,

$$\begin{aligned} B \wedge C &= \bigwedge \{(b \wedge c) \mid B \subseteq (b) \text{ and } C \subseteq (c)\} \\ &= \bigvee \{(x \wedge y) \mid (x) \subseteq B \text{ and } (y) \subseteq C\}. \end{aligned}$$

Since \mathcal{Q}_e is a complete lattice, a well-defined binary operation is obtained as follows: for $M, N \in L_e$,

$$M \rightarrow N = \bigwedge \{(m \rightarrow n) \mid (m) \subseteq M \text{ and } N \subseteq (n)\}.$$

The binary operation satisfies the definition of a relative pseudocomplement, since $Z = \bigvee \{(z) \mid (z) \subseteq Z\} \subseteq M \rightarrow N$ iff, for every $(m) \subseteq M, N \subseteq (n)$ and $(z) \subseteq Z, z \leq m \rightarrow n$ iff, for every $(m) \subseteq M, N \subseteq (n)$ and $(z) \subseteq Z, z \wedge m \leq n$ iff $Z \wedge M = \bigvee \{(z \wedge m) \mid (z) \subseteq Z \text{ and } (m) \subseteq M\} \subseteq \bigwedge \{(n) \mid N \subseteq (n)\} = N$.

Thus \mathcal{Q}_e is an implicative lattice.

Furthermore, the embedding $x \rightarrow (x)$ is an implicative homomorphism, since, for every $a, b \in L$, $(a) \rightarrow (b) = \bigwedge \{(x \rightarrow y) \mid (x) \subseteq (a) \text{ and } (b) \subseteq (y)\}$; but, by (1.3), $x \leq a$ implies $a \rightarrow b \leq x \rightarrow b$ and $b \leq y$ implies $x \rightarrow b \leq x \rightarrow y$; hence $a \rightarrow b \leq x \rightarrow y$; thus $(a) \rightarrow (b) = (a \rightarrow b)$. \square

COROLLARY 4.1.a. *For every $b \in L$ and $X \in L_e$,*

$$X \rightarrow (b) = \bigwedge \{(x \rightarrow b) \mid x \in X\} \text{ and } (b) \rightarrow X = \bigwedge \{(b \rightarrow y) \mid X \subseteq (y)\}.$$

PROOF. By the preceding theorem,

$$X \rightarrow (b) = \bigwedge \{(x \rightarrow y) \mid (x) \subseteq X \text{ and } (b) \subseteq (y)\},$$

but $(b) \subseteq (y)$ if and only if $b \leq y$ which implies that $x \rightarrow b \leq x \rightarrow y$; also $X \in L_e$ if and only if $X = Y^+$ for some $Y \subseteq L$ so $(x) \subseteq X$ if and only if $x \in X$; hence $X \rightarrow (b) = \bigwedge \{(x \rightarrow b) \mid x \in X\}$. Similarly, $(b) \rightarrow X = \bigwedge \{(z \rightarrow y) \mid (z) \subseteq (b) \text{ and } X \subseteq (y)\} = \bigwedge \{(b \rightarrow y) \mid X \subseteq (y)\}$ since $(z) \subseteq (b)$ if and only if $z \leq b$ which implies $b \rightarrow y \leq z \rightarrow y$. \square

COROLLARY 4.1.b. *If m is a meet-irreducible element of \mathcal{Q} then (m) is a meet-irreducible element of \mathcal{Q}_e .*

PROOF. For any $X \in L_e$ such that $X \not\subseteq (m)$ there exists $x_0 \in X$ such that $x_0 \not\leq m$ so if m is a meet-irreducible element of L then, by Theorem 2.1, $x_0 \rightarrow m = m$, and, by Corollary 4.1.a, $X \rightarrow (m) = \bigwedge \{(x \rightarrow m) \mid x \in X\}$ so $X \rightarrow (m) \subseteq (x_0 \rightarrow m) = (m)$; but, by (1.1), $(m) \subseteq X \rightarrow (m)$; therefore $X \rightarrow (m) = (m)$ and (m) is a meet-irreducible element of \mathcal{Q}_e . \square

COROLLARY 4.1.c. *If every element of an implicative lattice has a meet-irreducible decomposition then every element of its MacNeille completion has a meet-irreducible decomposition.*

COROLLARY 4.1.d. *If a is an atom of \mathcal{Q} , then (a) is an atom of \mathcal{Q}_e .*

PROOF. For any $Y \in L_e$ such that $(a] \not\subseteq Y = Z^+$ there exists $z_0 \in Z$ such that $a \not\leq z_0$, so if a is an atom of \mathcal{Q} , then, by Theorem 2.2, $a \rightarrow z_0 = a^*$ and, by Corollary 4.1.a, $(a] \rightarrow Y \subseteq (a \rightarrow z_0) = (a^*]$; but $(a^*] \subseteq (a] \rightarrow Y$ since $(a^*] \wedge (a] = (0] \subseteq Y$; therefore $(a] \rightarrow Y = (a^*] = (a]^*$ and $(a]$ is an atom of \mathcal{Q}_e . \square

COROLLARY 4.1.e. *The MacNeille completion of an atomic implicative lattice is atomic.*

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