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MEET-IRREDUCIBLE ELEMENTS IN IMPLICATIVE LATTICES[1]

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ABSTRACT. A characterization of meet-irreducible elements and atoms in an implicative lattice is obtained and used to derive the following theorems. A complete lattice is implicative and every element has a meet-irreducible decomposition if and only if there are enough principal prime relative annihilator ideals to separate distinct elements. The MacNeille completion of an implicative lattice is an implicative lattice; furthermore the embedding preserves relative pseudocomplements, meet-irreducible elements and atoms.

1. **Introduction.** An *implicative lattice* is a system $\mathfrak{L}=(L, \leqq, \vee, \wedge, \rightarrow, 1)$ such that $(L, \vee, \wedge)$ is a lattice and $\rightarrow$ is a binary operation on L which, for every $a, b, c \in L$, satisfies

$$c \leqq a \rightarrow b \quad \text{if and only if} \quad c \wedge a \leqq b.$$

The element $a \rightarrow b$ is called the *pseudocomplement of a relative to b*. The variety of implicative lattices and its relationship to logic and other mathematical systems has been extensively studied **[2]**, **[8]**, **[9]**, **[11]**, **[12]** with considerable lack of uniformity in terminology and notation. The following is a brief list of properties for later reference.

If $\mathfrak{L}$ is an implicative lattice and $x, y, z \in L$, then

(1.1) $y \leqq x \rightarrow y$;

(1.2) $x \leqq y$ if and only if $x \rightarrow y = 1$;

(1.3) $x \leqq y$ implies $z \rightarrow x \leqq z \rightarrow y$ and $y \rightarrow z \leqq x \rightarrow z$;

(1.4) $(y \wedge z) \rightarrow x = z \rightarrow (y \rightarrow x) = y \rightarrow (z \rightarrow x) = (z \rightarrow y) \rightarrow (z \rightarrow x)$;

(1.5) L is a distributive lattice;

(1.6) if $B \subseteq L$ such that $\bigwedge B$ exists then, for every $a \in L$, $\bigwedge \{a \rightarrow y \mid y \in B\}$ exists and equals $a \rightarrow \bigwedge B$ **[11**, p. 136].

2. **Meet-irreducible elements and atoms.** In 1938, Ward **[13]** obtained a necessary condition for an element to be meet-irreducible in a residuated

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lattice. The following theorem shows this condition to be both necessary and sufficient for an element to be meet-irreducible in an implicative lattice.

THEOREM 2.1. *An element m is meet-irreducible in an implicative lattice $\mathfrak{L}$ if and only if $x\nleqq m$ implies $x\to m=m$.*

PROOF OF NECESSITY. Let m be a meet-irreducible element of an implicative lattice $\mathfrak{L}$ and $x\in L$ such that $x\nleqq m$. By definition of $x\to m$ it will suffice to show that $x\wedge z\leqq m$ implies $z\leqq m$, since clearly $z\leqq m$ implies $x\wedge z\leqq m$. If $x\wedge z\leqq m$ then $m=m\vee(x\wedge z)=(m\vee x)\wedge(m\vee z)$, so $m=m\vee x$ or $m=m\vee z$ since m is meet-irreducible, but $x\nleqq m$ implies $m\neq m\vee x$ thus $m=m\vee z$, i.e. $z\leqq m$.

PROOF OF SUFFICIENCY. Let $m\in L$ such that, for every $x\nleqq m$, $x\to m=m$ and $p, q\in L$ such that $m<p$ and $m=p\wedge q$. Then $p\nleqq m$ so $p\to m=m=p\wedge q$, so by definition of $p\to m$ $q\leqq m$, but also $m=p\wedge q\leqq q$, hence $m=q$; therefore m is meet-irreducible. □

COROLLARY 2.1.a. *An element m is meet-irreducible in a Boolean algebra if and only if $x\nleqq m$ implies $m'\leqq x$.*

COROLLARY 2.1.b. *If b is meet-irreducible then, for every a, $a\to b$ is meet-irreducible.*

PROOF. If $x\nleqq a\to b$ then $x\nleqq b\leqq a\to b$ so $x\to b=b$; hence $x\to(a\to b)=a\to(x\to b)=a\to b$, by (1.4). □

COROLLARY 2.1.c. *If $\tau:L_1\to L_2$ is an implicative homomorphism and m is a meet-irreducible element of L_1 then $m\tau$ is a meet-irreducible element of $L_1\tau$.*

PROOF. If $x\tau\in L_1\tau$ such that $x\tau\nleqq m\tau$ then $x\nleqq m$ hence $x\to m=m$ and $(x\to m)\tau=x\tau\to m\tau=m\tau$. □

COROLLARY 2.1.d. *If M is a set of meet-irreducible elements and $b=\bigwedge M$ then, for every a, $a\to b=\bigwedge\{a\to m \mid m\in M\}=\bigwedge\{m \mid m\in M \text{ and } a\nleqq m\}$.*

Let $\mathfrak{L}=(L,\vee,\wedge,\to,1,0)$ be a *pseudo-Boolean algebra*, i.e. an implicative lattice with least element, 0, and $x^*=x\to 0$ denote the *pseudocomplement of x*.

THEOREM 2.2. *An element a is an atom in a pseudo-Boolean algebra if and only if $a\neq 0$ and $a\nleqq y$ implies $a\to y=a^*$.*

PROOF OF NECESSITY. Let a be an atom and $y\in L$ such that $a\nleqq y$. It will suffice to show that $z\leqq a^*$ if and only if $z\wedge a\leqq y$. If $z\leqq a^*$ then $z\wedge a=0$; so, for every y, $z\wedge a\leqq y$. Conversely, if $z\wedge a\leqq y$ then since a is an atom $z\wedge a=0$ or $z\wedge a=a$, but $a=z\wedge a\leqq y$ contradicts $a\nleqq y$; thus $0=z\wedge a$, i.e. $z\leqq a^*$.

PROOF OF SUFFICIENCY. Let $a\neq 0$, for every y, $a\nleqq y$ implies $a\to y=a^*$, and $z<a$. Then $z=z\wedge a$ and $a\nleqq z$ so $z\leqq a\to z=a^*$; hence $z=z\wedge a=0$, i.e. a is an atom. □

COROLLARY 2.2.a. *If a is an atom then a^* is meet-irreducible.*

PROOF. First note that $x\nleqq a^*$ if and only if $a\nleqq x^*$; so if a is an atom and $x\nleqq a^*$ then $a\to x^*=a^*$, so by (1.4), $x\to a^*=x\to(a\to x^*)=a\to(x\to x^*)=a\to x^*=a^*$; thus a^* is meet-irreducible by Theorem 2.1. □

3. **Meet-irreducible decompositions and relative annihilator ideals.** Balachandran [**1**, p. 550] has shown that a complete lattice is implicative if every element has a finite meet-prime decomposition and Newman [**10**, p. 31] that a complete distributive lattice is meet-continuous if every element has a finite meet-irreducible decomposition. The next theorem generalizes these results.

THEOREM 3.1. *If every element of a complete distributive lattice has a meet-irreducible decomposition, then the lattice is implicative.*

PROOF. Let $\mathfrak{L}$ be a complete distributive lattice in which every element has a meet-irreducible decomposition, M be the set of meet-irreducible elements and, for each $x\in L$, $M(x)=\{m\in M \mid x\leqq m\}$. Since $\mathfrak{L}$ is complete, a well-defined binary operation is obtained by defining, for every $a, b\in L$, $a\to b=\bigwedge\,[M(b)-M(a)]$. Observe that, for every $x\in L$, $1\in M(x)$, so $a\to b=1$ if and only if $M(b)-M(a)=\varnothing$ if and only if $a\leqq b$.

McKinsey and Tarski [**8**, Theorem 1.4] have shown that $\mathfrak{L}$ is implicative provided, for every a, b, $c\in L$, (1) $a\wedge a\to b\leqq b$, (2) $a\leqq b\to(a\wedge b)$, and (3) $c\to(a\wedge b)\leqq c\to a$. First note that $a\wedge a\to b\leqq b$ is implied by $M(b)\subseteq M(a)\cup[M(b)-M(a)]$, so (1) is established. To establish (2), we use the distributivity to show that $M(a\wedge b)=M(a)\cup M(b)$; for if $q\in M(a\wedge b)$ then $q=(a\wedge b)\vee q=(a\vee q)\wedge(b\vee q)$, so $q=a\vee q$ or $q=b\vee q$, i.e. $q\in M(a)\cup M(b)$; from this it follows that $M(a\wedge b)-M(b)\subseteq M(a)$, so $a=\bigwedge M(a)\leqq\bigwedge\,[M(a\wedge b)-M(b)]=b\to(a\wedge b)$. Finally, $M(a)\subseteq M(a\wedge b)$, so $M(a)-M(c)\subseteq M(a\wedge b)-M(c)$ and $c\to(a\wedge b)=\bigwedge\,[M(c)-M(a\wedge b)]\leqq\bigwedge\,[M(a)-M(c)]=c\to a$. □

Mandelker [7] introduced the notion of the *annihilator of a relative to* $b=\langle a, b\rangle=\{z \mid z\wedge a\leqq b\}$ and showed that a lattice is distributive if and only if every relative annihilator is an ideal. Clearly, a lattice is implicative if and only if every relative annihilator is a principal ideal, in which case $\langle a, b\rangle=(a\to b]$. Note further that in a distributive lattice there are enough relative annihilator ideals to separate distinct elements, since $a\nleqq b$ implies $b\in\langle a, b\rangle$ but $a\notin\langle a, b\rangle$. Also the relative annihilator $\langle a, b\rangle$ is proper if and only if $a\nleqq b$, so the class of relative annihilators needed to separate elements need contain none for which $a\leqq b$.

THEOREM 3.2. *Let $\mathfrak{L}$ be a complete lattice. Then $\mathfrak{L}$ is an implicative lattice in which every element has a meet-irreducible decomposition if and only if there are enough principal prime relative annihilator ideals to separate distinct elements.*

PROOF. If there are enough principal prime relative annihilator ideals to separate elements then there are enough prime ideals to separate elements; hence L is distributive and the principal ideal $(p]$ is prime if and only if p is meet-irreducible. For each $x \in L$, let $M(x)=\{m \in L \mid m$ is meet-irreducible and $x \leqq m\}$, so $\bigwedge M(x)=\bar{x}$ exists in L and $x \leqq \bar{x}$; but if $x<\bar{x}$ then there exists a principal prime ideal $(p]$ such that $x \in (p]$ but $\bar{x} \notin (p]$; hence there exists a meet-irreducible element p such that $p \in M(x)$ but $\bar{x} \nleqq p$ contradicting $\bar{x}=M(x)$; hence $x=\bigwedge M(x)$, and, by Theorem 3.1, $\mathfrak{L}$ is implicative.

Conversely, suppose L is an implicative lattice in which every element has a meet-irreducible decomposition and $b=\bigwedge M(b) \nleqq a=\bigwedge M(a)$. Then, for some $\bar{m} \in M(a)$, $\bar{m} \notin M(b)$, so $b \nleqq \bar{m}$ and $a \leqq \bar{m} \leqq b \rightarrow \bar{m}$; thus $a \in \langle b, \bar{m}\rangle=(\bar{m}]$ and $b \notin \langle b, \bar{m}\rangle=(\bar{m}]$.

4. **The MacNeille completion.** MacNeille [**6**] showed that by generalizing Dedekind's completion by cuts of the rational numbers, any partially ordered set can be embedded in a complete lattice in a manner which preserves the order relation and any existing greatest lower bounds and least upper bounds (i.e. it is a regular embedding). Furthermore, the embedding is both meet and join dense. The embedding does not necessarily preserve other important lattice structure; Funayama [**4**] constructed a distributive lattice whose MacNeille completion was not even modular, and Crawley [**3**] constructed a distributive lattice which could not be regularly embedded in any complete modular lattice. The existence of complements or relative pseudocomplements seems however to be helpful in preserving distributivity, as Glivenko [**5**] and Stone [**12**] show that the MacNeille completion of a Boolean algebra is a Boolean algebra and we now show that the MacNeille completion of an implicative lattice is an implicative lattice; furthermore the embedding preserves relative pseudocomplements and any existing meet-irreducible elements and atoms.

THEOREM 4.1. *The MacNeille completion of an implicative lattice is an implicative lattice, and the embedding preserves relative pseudocomplements.*

PROOF. Let $\mathfrak{L}$ be an implicative lattice and $\mathfrak{L}_e$ denote its MacNeille completion. Recall [**2**, Chapter V, §9] that $L_e=\{A^+ \mid A \subseteq L\}$ where A^+ is the set of lower bounds of A in $\mathfrak{L}$, set inclusion is the partial order and set intersection is the meet operation, and the embedding is $x \rightarrow (x]$, the principal ideal generated by x. It is easily shown that, for every $B, C \in L_e$,

$$\begin{aligned} B \wedge C &= \bigwedge \{(b \wedge c] \mid B \subseteq (b] \text{ and } C \subseteq (c]\} \\ &= \bigvee \{(x \wedge y] \mid (x] \subseteq B \text{ and } (y] \subseteq C\}. \end{aligned}$$

Since $\mathfrak{L}_e$ is a complete lattice, a well-defined binary operation is obtained as follows: for $M, N \in L_e$,

$$M \to N = \bigwedge \{(m \to n] \mid (m] \subseteq M \text{ and } N \subseteq (n]\}.$$

The binary operation satisfies the definition of a relative pseudocomplement, since $Z = \bigvee \{(z] | (z] \subseteq Z\} \subseteq M \to N$ iff, for every $(m] \subseteq M$, $N \subseteq (n]$ and $(z] \subseteq Z$, $z \leqq m \to n$ iff, for every $(m] \subseteq M$, $N \subseteq (n]$ and $(z] \subseteq Z$, $z \wedge m \leqq n$ iff

$$Z \wedge M = \bigvee \{(z \wedge m] \mid (z] \subseteq Z \text{ and } (m] \subseteq M\} \subseteq \bigwedge \{(n] \mid N \subseteq (n]\} = N.$$

Thus $\mathfrak{L}_e$ is an implicative lattice.

Furthermore, the embedding $x \to (x]$ is an implicative homomorphism, since, for every $a, b \in L$, $(a] \to (b] = \bigwedge \{(x \to y] | (x] \subseteq (a]$ and $(b] \subseteq (y]\}$; but, by (1.3), $x \leqq a$ implies $a \to b \leqq x \to b$ and $b \leqq y$ implies $x \to b \leqq x \to y$; hence $a \to b \leqq x \to y$; thus $(a] \to (b] = (a \to b]$. □

COROLLARY 4.1.a. *For every $b \in L$ and $X \in L_e$,*

$$X \to (b] = \bigwedge \{(x \to b] \mid x \in X\} \text{ and } (b] \to X = \bigwedge \{(b \to y] \mid X \subseteq (y]\}.$$

PROOF. By the preceding theorem,

$$X \to (b] = \bigwedge \{(x \to y] \mid (x] \subseteq X \text{ and } (b] \subseteq (y]\},$$

but $(b] \subseteq (y]$ if and only if $b \leqq y$ which implies that $x \to b \leqq x \to y$; also $X \in L_e$ if and only if $X = Y^+$ for some $Y \subseteq L$ so $(x] \subseteq X$ if and only if $x \in X$; hence $X \to (b] = \bigwedge \{(x \to b] | x \in X\}$. Similarly, $(b] \to X = \bigwedge \{(z \to y] | (z] \subseteq (b]$ and $X \subseteq (y]\} = \bigwedge \{(b \to y] | X \subseteq (y]\}$ since $(z] \subseteq (b]$ if and only if $z \leqq b$ which implies $b \to y \leqq z \to y$. □

COROLLARY 4.1.b. *If m is a meet-irreducible element of $\mathfrak{L}$ then $(m]$ is a meet-irreducible element of $\mathfrak{L}_e$.*

PROOF. For any $X \in L_e$ such that $X \nsubseteq (m]$ there exists $x_0 \in X$ such that $x_0 \nleqq m$ so if m is a meet-irreducible element of L then, by Theorem 2.1, $x_0 \to m = m$, and, by Corollary 4.1.a, $X \to (m] = \bigwedge \{(x \to m] | x \in X\}$ so $X \to (m] \subseteq (x_0 \to m] = (m]$; but, by (1.1), $(m] \subseteq X \to (m]$; therefore $X \to (m] = (m]$ and $(m]$ is a meet-irreducible element of $\mathfrak{L}_e$. □

COROLLARY 4.1.c. *If every element of an implicative lattice has a meet-irreducible decomposition then every element of its MacNeille completion has a meet-irreducible decomposition.*

COROLLARY 4.1.d. *If a is an atom of $\mathfrak{L}$, then $(a]$ is an atom of $\mathfrak{L}_e$.*

PROOF. For any $Y\in L_e$ such that $(a]\nsubseteq Y=Z^+$ there exists $z_0\in Z$ such that $a\nleq z_0$, so if a is an atom of $\mathfrak{L}$, then, by Theorem 2.2, $a\to z_0=a^*$ and, by Corollary 4.1.a, $(a]\to Y\subseteq(a\to z_0]=(a^*]$; but $(a^*]\subseteq(a]\to Y$ since $(a^*]\wedge(a]=(0]\subseteq Y$; therefore $(a]\to Y=(a^*]=(a]^*$ and $(a]$ is an atom of $\mathfrak{L}_e$. □

COROLLARY 4.1.e. *The MacNeille completion of an atomic implicative lattice is atomic.*

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