PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, July 1972

THE NUMBER OF SQUARE-FULL DIVISORS OF AN INTEGER

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ABSTRACT. Let $\alpha(n)$ denote the number of square-full divisors of n. In this note an asymptotic formula for $\sum_{n\leqq x}\alpha(n)$ is established.

1. **Introduction.** A positive integer $n>1$ is called square-full if in the canonical representation of n into prime powers each prime occurs with multiplicity at least two, or equivalently, if p is a prime dividing n, then p^2 also divides n. The integer 1 is also considered to be square-full. Let L denote the set of square-full integers. A divisor $d>0$ of the positive integer n is called square-full if $d\in L$. Let $\alpha(n)$ denote the number of square-full divisors of n. It is clear that $\alpha(1)=1$ and if $1<n=\prod_{i=1}^{r}p_i^{a_i}$, then $\alpha(n)=\prod_{a_i\geqq 2}(a_i)$, since every square-full divisor of n is a term in the expansion of the product $\prod_{a_i\geqq 2}(1+p_i^2+\cdots+p_i^{a_i})$ and conversely.

Let $\beta(n)$ denote the number of divisors $d>0$ of the positive integer n such that $\gamma(d)=\gamma(n)$, where $\gamma(n)$ is the core of n, that is, the maximal square-free divisor of n. It is clear that $\beta(1)=1$ and if $1<n=\prod_{i=1}^{r}p_i^{a_i}$, then $\beta(n)=\prod_{i=1}^{r}(a_i)$, so that $\beta(n)$ is the same as $\alpha(n)$.

The object of the present note is to establish an asymptotic formula for $\sum_{n\leqq x}\alpha(n)$. In fact, we prove

$$(1.1)\qquad \sum_{n\leqq x}\alpha(n)=\frac{\zeta(2)\zeta(3)}{\zeta(6)}x+\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)}x^{1/2}+\frac{\zeta(\frac{1}{3})\zeta(\frac{2}{3})}{\zeta(2)}x^{1/3}+O(x^{\theta}),$$

where $\zeta(s)$ is the Riemann zeta function defined for $s>0$, $s\neq 1$, by

$$(1.2)\qquad \zeta(s)=\frac{s}{s-1}-s\int_1^{\infty}\frac{(t-[t])}{t^{s+1}}\,dt,$$

and θ is the number which appears in the divisor problem, viz.,

$$(1.3)\qquad \sum_{n_1n_2^2n_3^3\leqq x}1=\zeta(2)\zeta(3)x+\zeta(\tfrac{1}{2})\zeta(\tfrac{3}{2})x^{1/2}+\zeta(\tfrac{1}{3})\zeta(\tfrac{2}{3})x^{1/3}+O(x^{\theta}).$$

Received by the editors November 10, 1971.
AMS 1970 *subject classifications*. Primary 10H25.
Key words and phrases. Square-full divisors, Riemann zeta function.

It is known that $\frac{1}{6}<\theta\leqq 20/69$. The left side inequality for θ is due to E. Landau (cf. [**2**], also cf. [**3**]) and the right side inequality for θ is the recent result due to E. Krätzel (cf. [**1**, Satz. 6]).

2. **Proof of (1.1).** We have $\alpha(n)=\sum_{d\delta=n;d\in L} 1$. Since $d\in L$, d can be uniquely represented in the form $d=a^2b^3$, where b is square-free. Hence $\alpha(n)=\sum_{d\delta=n}\sum_{a^2b^3=d}\mu^2(b)$, where μ(b) is the Möbius function. Since $\mu^2(b)=\sum_{e^2f=b}\mu(e)$, we have

$$\alpha(n)=\sum_{d\delta=n}\sum_{a^2e^6f^3=d}\mu(e)=\sum_{a^2e^6f^3\delta=n}\mu(e).$$

Hence

$$\begin{aligned}\sum_{n\leqq x}\alpha(n) &= \sum_{a^2e^6f^3\delta\leqq x}\mu(e)=\sum_{e\leqq x^{1/6}}\mu(e)\sum_{\delta a^2f^3\leqq x/e^6}1\\ &=\sum_{e\leqq x^{1/6}}\mu(e)\left\{\zeta(2)\zeta(3)\frac{x}{e^6}+\zeta(\tfrac{1}{2})\zeta(\tfrac{3}{2})\frac{x^{1/2}}{e^3}+\zeta(\tfrac{1}{3})\zeta(\tfrac{2}{3})\frac{x^{1/3}}{e^2}+O\left(\frac{x^\theta}{e^{6\theta}}\right)\right\}\\ &=\zeta(2)\zeta(3)x\sum_{e\leqq x^{1/6}}\frac{\mu(e)}{e^6}+\zeta(\tfrac{1}{2})\zeta(\tfrac{3}{2})x^{1/2}\sum_{e\leqq x^{1/6}}\frac{\mu(e)}{e^3}\\ &\quad+\zeta(\tfrac{1}{3})\zeta(\tfrac{2}{3})x^{1/3}\sum_{e\leqq x^{1/6}}\frac{\mu(e)}{e^2}+O\left(x^\theta\sum_{e\leqq x^{1/6}}\frac{1}{e^{6\theta}}\right)\\ &=\zeta(2)\zeta(3)x\left\{\frac{1}{\zeta(6)}+O(x^{-5/6})\right\}\\ &\quad+\zeta(\tfrac{1}{2})\zeta(\tfrac{3}{2})x^{1/2}\{1/\zeta(3)+O(x^{-1/3})\}\\ &\quad+\zeta(\tfrac{1}{3})\zeta(\tfrac{2}{3})x^{1/3}\{1/\zeta(2)+O(x^{-1/6})\}+O(x^\theta),\end{aligned}$$

since $6\theta>1$.

Hence

$$\sum_{n\leqq x}\alpha(n)=\frac{\zeta(2)\zeta(3)}{\zeta(6)}x+\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)}x^{1/2}+\frac{\zeta(\frac{1}{3})\zeta(\frac{2}{3})}{\zeta(2)}x^{1/3}+O(x^\theta),$$

so that (1.1) follows.

References

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