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# THE NUMBER OF SQUARE-FULL DIVISORS OF AN INTEGER

D. SURYANARAYANA AND R. SITA RAMA CHANDRA RAO

ABSTRACT. Let $\alpha(n)$ denote the number of square-full divisors of $n$. In this note an asymptotic formula for $\sum_{n\leqq x}\alpha(n)$ is established.

1. **Introduction.** A positive integer $n>1$ is called square-full if in the canonical representation of $n$ into prime powers each prime occurs with multiplicity at least two, or equivalently, if $p$ is a prime dividing $n$, then $p^2$ also divides $n$. The integer 1 is also considered to be square-full. Let $L$ denote the set of square-full integers. A divisor $d>0$ of the positive integer $n$ is called square-full if $d\in L$. Let $\alpha(n)$ denote the number of square-full divisors of $n$. It is clear that $\alpha(1)=1$ and if $1<n=\prod_{i=1}^{r} p_i^{a_i}$, then $\alpha(n)=\prod_{a_i\geqq 2}(a_i)$, since every square-full divisor of $n$ is a term in the expansion of the product $\prod_{a_i\geqq 2}(1+p_i^2+\cdots+p_i^{a_i})$ and conversely.

Let $\beta(n)$ denote the number of divisors $d>0$ of the positive integer $n$ such that $\gamma(d)=\gamma(n)$, where $\gamma(n)$ is the core of $n$, that is, the maximal square-free divisor of $n$. It is clear that $\beta(1)=1$ and if $1<n=\prod_{i=1}^{r} p_i^{a_i}$, then $\beta(n)=\prod_{i=1}^{r}(a_i)$, so that $\beta(n)$ is the same as $\alpha(n)$.

The object of the present note is to establish an asymptotic formula for $\sum_{n\leqq x}\alpha(n)$. In fact, we prove

$$(1.1)\qquad \sum_{n\leqq x}\alpha(n)=\frac{\zeta(2)\zeta(3)}{\zeta(6)}x+\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)}x^{1/2}+\frac{\zeta(\frac{1}{3})\zeta(\frac{2}{3})}{\zeta(2)}x^{1/3}+O(x^{\theta}),$$

where $\zeta(s)$ is the Riemann zeta function defined for $s>0$, $s\neq 1$, by

$$(1.2)\qquad \zeta(s)=\frac{s}{s-1}-s\int_1^{\infty}\frac{(t-[t])}{t^{s+1}}\,dt,$$

and $\theta$ is the number which appears in the divisor problem, viz.,

$$(1.3)\qquad \sum_{n_1 n_2^2 n_3^3\leqq x}1=\zeta(2)\zeta(3)x+\zeta(\tfrac{1}{2})\zeta(\tfrac{3}{2})x^{1/2}+\zeta(\tfrac{1}{3})\zeta(\tfrac{2}{3})x^{1/3}+O(x^{\theta}).$$

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It is known that $\frac{1}{6}<\theta\leqq 20/69$. The left side inequality for $\theta$ is due to E. Landau (cf. [2], also cf. [3]) and the right side inequality for $\theta$ is the recent result due to E. Krätzel (cf. [1, Satz. 6]).

2. **Proof of (1.1).** We have $\alpha(n)=\sum_{d\delta=n;d\in L}1$. Since $d\in L$, $d$ can be uniquely represented in the form $d=a^2b^3$, where $b$ is square-free. Hence $\alpha(n)=\sum_{d\delta=n}\sum_{a^2b^3=d}\mu^2(b)$, where $\mu(b)$ is the Möbius function. Since $\mu^2(b)=\sum_{e^2f=b}\mu(e)$, we have

$$\alpha(n)=\sum_{d\delta=n}\sum_{a^2e^6f^3=d}\mu(e)=\sum_{a^2e^6f^3\delta=n}\mu(e).$$

Hence

$$\begin{aligned}\sum_{n\leqq x}\alpha(n) &= \sum_{a^2e^6f^3\delta\leqq x}\mu(e)=\sum_{e\leqq x^{1/6}}\mu(e)\sum_{\delta a^2f^3\leqq x/e^6}1\\
&=\sum_{e\leqq x^{1/6}}\mu(e)\left\{\zeta(2)\zeta(3)\frac{x}{e^6}+\zeta(\tfrac{1}{2})\zeta(\tfrac{3}{2})\frac{x^{1/2}}{e^3}+\zeta(\tfrac{1}{3})\zeta(\tfrac{2}{3})\frac{x^{1/3}}{e^2}+O\left(\frac{x^\theta}{e^{6\theta}}\right)\right\}\\
&=\zeta(2)\zeta(3)x\sum_{e\leqq x^{1/6}}\frac{\mu(e)}{e^6}+\zeta(\tfrac{1}{2})\zeta(\tfrac{3}{2})x^{1/2}\sum_{e\leqq x^{1/6}}\frac{\mu(e)}{e^3}\\
&\quad+\zeta(\tfrac{1}{3})\zeta(\tfrac{2}{3})x^{1/3}\sum_{e\leqq x^{1/6}}\frac{\mu(e)}{e^2}+O\left(x^\theta\sum_{e\leqq x^{1/6}}\frac{1}{e^{6\theta}}\right)\\
&=\zeta(2)\zeta(3)x\left\{\frac{1}{\zeta(6)}+O(x^{-5/6})\right\}\\
&\quad+\zeta(\tfrac{1}{2})\zeta(\tfrac{3}{2})x^{1/2}\{1/\zeta(3)+O(x^{-1/3})\}\\
&\quad+\zeta(\tfrac{1}{3})\zeta(\tfrac{2}{3})x^{1/3}\{1/\zeta(2)+O(x^{-1/6})\}+O(x^\theta),\end{aligned}$$

since $6\theta>1$.

Hence

$$\sum_{n\leqq x}\alpha(n)=\frac{\zeta(2)\zeta(3)}{\zeta(6)}x+\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)}x^{1/2}+\frac{\zeta(\frac{1}{3})\zeta(\frac{2}{3})}{\zeta(2)}x^{1/3}+O(x^\theta),$$

so that (1.1) follows.

## References

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DEPARTMENT OF MATHEMATICS, ANDHRA UNIVERSITY, WALTAIR, INDIA