PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, July 1972

COHERENCE OF POLYNOMIAL RINGS OVER SEMISIMPLE ALGEBRAIC ALGEBRAS

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ABSTRACT. It is shown that polynomial rings in finitely or infinitely many central indeterminates, over a commutative algebraic algebra without nilpotent elements, are coherent. If the coefficient ring is algebraic over the real numbers, then the commutativity assumption, above, may be dropped.

In this paper all rings have identity, all modules are unital, and all ring homomorphisms preserve the identity.

DEFINITION 1. A ring R is *left coherent* if, for each finitely generated left ideal I in R, there exists an exact sequence of left R-modules

$$0 \to K \to F \to I \to 0$$

such that F and K are finitely generated and F is free.

Right coherent rings may be similarly defined. The concept of a coherent ring was introduced by Chase in [**4**]. He showed, in [**4**, Theorem 2.1], that a ring R is **left** coherent if and only if the direct product of any family of flat **right** R-modules is flat. As left Noetherian rings are clearly left coherent, this suggests that left coherent rings are, at least with respect to homological properties, a generalization of left Noetherian rings. This raises the following question: If R is an arbitrary left coherent ring, is the polynomial ring $R[Z]$ left coherent too? Soublin, in [**10**], answered this question in the negative, even for commutative R. However, he showed in [**9**, Theorems 21 and 22] that if R is commutative and von Neumann regular (i.e. for each $r \in R$ there exists $r' \in R$ such that $rr'r=r$), then $R[Z]$ is coherent and its finitely generated ideals are principal.

In this paper we prove the following result:

THEOREM. *Let A be a central algebraic algebra, without nilpotent elements, over some field L. Suppose that at least one of the following two*

Received by the editors August 6, 1971.

AMS 1969 *subject classifications*. Primary 1320, 1620; Secondary 1656, 1644.

Key words and phrases. Commutative semisimple algebraic algebras, polynomial rings, coherent rings, topological representations, faithfully flat over rings, direct limits of coherent rings.

[1] The author was supported at the University of Hawaii by a National Research Council of Canada Postdoctorate fellowship while carrying out research for this paper.

hypothesis is satisfied:

(α) *A is commutative*,

(β) *L is the field of real numbers.*

Then the polynomial ring $A[\{Z_\alpha\}]$ is left and right coherent, for any finite or infinite set $\{Z_\alpha\}$ of central indeterminates. ▮

The above meanings for A, L, and $\{Z_\alpha\}$ are retained throughout this paper. By [**1**, Theorems 3.2 and 3.3], A is von Neumann regular. The proof of the above theorem requires the following lemmas and propositions.

PROPOSITION 1. *Let R be a ring. For $r \in R$ let $(0:r)$ denote $\{s \in R: sr=0\}$. Then R is left coherent if and only if*

(i) *for each $r \in R$, $(0:r)$ is finitely generated as a left ideal in R,*

(ii) *the intersection of any two finitely generated left ideals in R is again finitely generated.* ▮

PROPOSITION 2. *Let $\{R_\alpha\}$ be a directed system of* ***left*** *coherent rings such that, when $\alpha \leqq \beta$, R_β is flat as a* ***right*** *R_α-module. Then the direct limit of $\{R_\alpha\}$ is a* ***left*** *coherent ring.* ▮

DEFINITION 2. Let R be a subring of the ring S. Then S is *faithfully* ***right*** *flat over R* if

(i) S is flat as a ***right*** R-module,

(ii) if M is a ***left*** R-module such that $S \otimes_R M=0$, then $M=0$.

PROPOSITION 3. *Let R be a subring of the* ***left*** *coherent ring S, such that S is faithfully* ***right*** *flat over R. Then R is a* ***left*** *coherent ring.* ▮

Proposition 1 is part of [**4**, Theorem 2.2]. Proposition 2 is from [**2**, p. 63, Example 12]. Proposition 3 is [**5**, Corollary 2.1].

Notation. (i) We recall that a topological space is *Boolean* if it is compact, Hausdorff, and totally disconnected. A subset of a topological space X is *clopen* if it is both open and closed in X.

(ii) For any ring R and Boolean space X, let $\mathscr{C}(X, R)$ denote the ring of all continuous functions from X to R, where R has the discrete topology. In other words, $\mathscr{C}(X, R)$ is the ring of all locally constant functions from X to R. For $f \in \mathscr{C}(X, R)$ let $\ker(f)$ denote $\{x \in X: f(x)=0\}$. Clearly $\ker(f)$ is clopen in X.

(iii) If I is a left ideal in $\mathscr{C}(X, R)$ and $x \in X$, let I_x denote $\{f(x): f \in I\}$. Clearly I_x is a left ideal in R. It is easy to see, where J is also a left ideal in $\mathscr{C}(X, R)$, that $(I \cap J)_x = I_x \cap J_x$.

(iv) If A is commutative let F denote the algebraic closure of L. Otherwise let F denote the real quaternions.

Our results hinge upon the following topological representation of A.

PROPOSITION 4. *There is an embedding*

$$A \to \mathscr{C}(X, F),$$

for some Boolean space X.

PROOF. If A is commutative this is contained in [**1**, Theorem 6.1]. An alternate proof, if A is commutative, and the noncommutative case, occur in [**3**, Theorem 3.4 and concluding remark (d)]. ▌

Notation. Let B denote the ring $\mathscr{C}(X, F)$ from Proposition 4. Let $p \geqq 1$ be an integer. Let T denote the polynomial ring $F[Z_1, \cdots, Z_p]$. Clearly

$$B[Z_1, \cdots, Z_p] \cong \mathscr{C}(X, T).$$

LEMMA 1. *The ring* $B[Z_1, \cdots, Z_p]$ *is left coherent.*

PROOF. We shall use Proposition 1 to show that $\mathscr{C}(X, T)$ is left coherent.

First let f be an arbitrary element in $\mathscr{C}(X, T)$. Define a map $e: X \to T$ by $e(x)=1$ when $x \in \ker(f)$ and $e(x)=0$ when $x \notin \ker(f)$. The map e is continuous since $\ker(f)$ is clopen in X. Clearly e generates $(0:f)$.

Second, let I and J be two finitely generated left ideals in $\mathscr{C}(X, T)$. Suppose that $x \in X$. Since elements of $\mathscr{C}(X, T)$ are locally constant functions, there exists a neighborhood N_x of x such that, for $y \in N_x$, $I_y = I_x$ and $J_y = J_x$. Since T is left Noetherian, $I_x \cap J_x$ can be generated, for some integer $n(x) \geqq 1$, by elements $t_1(x), \cdots, t_{n(x)}(x)$ of T. Since $\{N_x : x \in X\}$ is an open cover of the Boolean space X, there is for some integer $m \geqq 1$, by [**8**, p. 12], a family $\{V_j : 1 \leqq j \leqq m\}$ of clopen subsets of X such that

(a) $\bigcup_{j=1}^m (V_j) = X$,

(b) $V_i \cap V_j = \varnothing$ if $i \neq j$,

(c) for each j, where $1 \leqq j \leqq m$, there exists $x(j) \in X$ such that $V_j \subseteq N_{x(j)}$.

Let $n = \sup\{n(x(j)) : 1 \leqq j \leqq m\}$. Whenever $n(x(j)) < i \leqq n$ set $t_i(x(j)) = 0$. For each i, where $1 \leqq i \leqq n$, define the map $h_i : X \to T$ by $h_i(x) = t_i(x(j))$ where $x \in V_j$. In view of (a)–(c) the h_i are well defined. They are in $\mathscr{C}(X, T)$ since the V_j are clopen. Let H be the ideal in $\mathscr{C}(X, T)$ generated by $\{h_i : 1 \leqq i \leqq n\}$. By construction $(I \cap J)_x = I_x \cap J_x = H_x$ for each $x \in X$. This establishes, via a compactness argument similar to the one given above, that $I \cap J = H$. Hence $I \cap J$ is finitely generated. The lemma now follows from Proposition 1. ▌

In view of Proposition 3 we could now establish that $A[Z_1, \cdots, Z_p]$ is left coherent by showing that $B[Z_1, \cdots, Z_p]$ is faithfully right flat over $A[Z_1, \cdots, Z_p]$.

LEMMA 2. *The ring B is faithfully right flat over A.*

PROOF. It is established in [**1**, Theorems 3.2 and 3.3] that A is a von Neumann regular ring. By [**7**, Proposition 4], such rings have the property that all of their modules, left and right, are flat. In particular, B is flat as a right A-module.

Let M be a left A-module. Suppose that $B \otimes_A M=0$. Then, using the flatness of M as a left A-module, we have

$$M \cong A \otimes_A M \subseteq B \otimes_A M = 0.$$

Thus $M=0$. ∎

LEMMA 3. *Suppose that the ring S is faithfully right flat over the subring R. Then $S[Z]$ is faithfully right flat over $R[Z]$.*

PROOF. First note that, as left S-modules, $S[Z] \cong S \otimes_R R[Z]$.

Next note that, for any left $S[Z]$-module M, there are the following left S-module isomorphisms:

$$S[Z] \otimes_{R[Z]} M \cong S \otimes_R R[Z] \otimes_{R[Z]} M \cong S \otimes_R M.$$

Thus, for any left $S[Z]$-modules M and N and homomorphism $f: M \to N$, the following diagram is commutative and its columns are isomorphisms, where f' and f^* are natural maps induced by f:

$$\begin{array}{ccc} f': S \otimes_R M & \longrightarrow & S \otimes_R N \\ \updownarrow & & \updownarrow \\ f^*: S[Z] \otimes_{R[Z]} M & \longrightarrow & S[Z] \otimes_{R[Z]} N. \end{array}$$

Since S is faithfully right flat over R it follows that if f is a monomorphism, then so are f' and (by the diagram) f^*. Thus $S[Z]$ is a flat right $R[Z]$-module. Similarly, if $S[Z] \otimes_{R[Z]} M=0$, then $S \otimes_R M=0$ so that $M=0$. ∎

LEMMA 4. *The ring $A[Z_1, \cdots, Z_p]$ is left coherent.*

PROOF. It follows from Lemma 2 and p applications of Lemma 3, that $B[Z_1, \cdots, Z_p]$ is faithfully right flat over $A[Z_1, \cdots, Z_p]$. The result now follows from Lemma 1 and Proposition 3. ∎

THEOREM. *Let $\{Z_\alpha\}$ be any set (finite or infinite) of central indeterminates. Then $A[\{Z_\alpha\}]$ is left coherent.*

PROOF. The family $\{A[Z_1, \cdots, Z_n]: n \geqq 1 \text{ and } \{Z_1, \cdots, Z_n\} \subseteq \{Z_\alpha\}\}$ is directed under inclusion. Clearly

$$\varinjlim(A[Z_1, \cdots, Z_n]) \cong A[\{Z_\alpha\}].$$

The theorem now follows from Lemma 4 and Proposition 2. ∎

REMARK. The definition of a nilpotent element is left-right symmetric. *Thus $A[\{Z_\alpha\}]$ is also a right coherent ring.*

COROLLARY. *Suppose that R is a ring such that, for each $r \in R$, there is an integer $m(r) \geqq 2$ satisfying*

$$r^{m(r)} = r. \tag{*}$$

Then $R[\{Z_\alpha\}]$ is a commutative (left) coherent ring, for any finite or infinite set $\{Z_\alpha\}$ of central indeterminates.

PROOF. It is a well-known result, due to Jacobson, that R is commutative. As in [**8**, Corollary 12.5] there exists a finite set of prime integers $\{p_1, \cdots, p_n\}$ and a ring direct sum decomposition of R,

$$R \cong R_1 \oplus \cdots \oplus R_n,$$

such that each R_i has characteristic p_i. Thus, as each R_i satisfies (*), it is an algebraic algebra without nilpotent elements over the field with p_i elements. Hence $R_i[\{Z_\alpha\}]$ is left coherent, for $1 \leqq i \leqq n$. Clearly

$$R[\{Z_\alpha\}] \cong R_1[\{Z_\alpha\}] \oplus \cdots \oplus R_n[\{Z_\alpha\}].$$

The result now follows from [**6**, Corollary 2.1] which states that a finite direct product of left coherent rings is again left coherent. ∎

REFERENCES

1. R. F. Arens and I. Kaplansky, *Topological representation of algebras*, Trans. Amer. Math. Soc. **63** (1948), 457–481. MR **10**, 7.

2. N. Bourbaki, *Eléments de mathématique*. Fasc. XXVII. *Algèbre commutative*. Chap. 1: *Modules plats*. Chap. 2: *Localisation*, Actualités Sci. Indust., no. 1290, Hermann, Paris, 1961. MR **36** #146.

3. A. Carson, *Representation of semi-simple algebraic algebras*, J. Algebra (to appear).

4. S. Chase, *Direct products of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473. MR **22** #11017.

5. M. Harris, *Some results on coherent rings*, Proc. Amer. Math. Soc. **17** (1966), 474–479. MR **33** #1332.

6. ———, *Some results on coherent rings*. II, Glasgow Math. J. **8** (1967), 123–126. MR **36** #6453.

7. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass., 1966. MR **34** #5857.

8. R. S. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc. No. 70 (1967). MR **36** #151.

9. J. Soublin, *Anneaux cohérents*, C. R. Acad. Sci. Paris Sér. A-B **267** (1968), A183–A186. MR **38** #1121.

10. ———, *Un anneau cohérent dont l'anneau des polynômes n'est pas cohérent*, C. R. Acad. Sci. Paris Sér. A-B **267** (1968), A241–A243, MR **38** #1123.

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