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A NONNORMAL OUTER FUNCTION IN H^p

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ABSTRACT. In this paper we construct an outer function in H^p for all p which is not normal.

The purpose of this note is to answer a question raised by Professor Joseph Cima; namely, we show the existence of nonnormal outer functions which are in H^p. See [1] for relevant definitions.

The existence of a function in H^p for all p which is not normal easily follows from a theorem of P. Lappan [2, p. 190] which says that given an unbounded function f, holomorphic on the unit disc D, then there exists a Blaschke product B such that Bf is not normal. However Bf is clearly not outer.

We shall construct an outer function with two different asymptotic values at $z=1$, which will imply by a theorem of O. Lehto and K. I. Virtanen that this function is not normal.

THEOREM [3, THEOREM 2, p. 53]. *Let f be meromorphic and normal in G, and let f have an asymptotic value α at a boundary point P along a Jordan curve lying in the closure of G. Then f possesses the angular limit α at the point P.*

We present the

EXAMPLE.

$$f(z) = \frac{1}{z}\operatorname{Log}\left(\frac{1+z}{1-z}\right)\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log t\,dt\right\} = \frac{1}{z}F_1F_2.$$

PROOF. $f\in H^p$ for all p since F_2 is bounded and F_1 is in H^p for all p. From the representation theorem of outer functions, we have that F_2 is outer. Furthermore, F_1 is a schlicht function and by Theorem 3.17 in [1, p. 51] the singular part of F_1 is identically one. Thus F_1/z is an outer function and hence f is an outer function. Finally, it is an elementary calculation to show that

$$\lim_{\theta\to 0^+}|f(e^{i\theta})| = 0 \quad\text{and}\quad \lim_{\theta\to 2\pi^-}|f(e^{i\theta})| = \infty.$$

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Thus by the above stated theorem, f is not normal and has all the desired properties.

An examination of the above example yields the following result:

LEMMA. *Given* $\psi: [0, 2\pi] \to R^+$ *with*

(a) $\psi \in L^p$,

(b) $\log \psi \in L^1$,

(c) $\lim_{t \to 0^+} \psi(t) = 0$,

(d) $\lim_{t \to 2\pi^-} \psi(t) = +\infty$,

then

$$\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{it}+z}{e^{it}-z}\right)\log \psi(t)\, dt\right\}$$

is an outer function in H^p *which is not normal.*

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