

PROPERTIES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FORM $y'' + a(t)b(y) = 0$

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ABSTRACT. Two theorems are presented which guarantee the boundedness and oscillation of solutions of certain classes of second order nonlinear differential equations.

In this paper we shall show boundedness of solutions of the differential equation $y'' + a(t)b(y) = 0$ with appropriate conditions on $a(t)$ and $b(y)$. Furthermore, all solutions will be oscillatory subject to a further condition. By oscillatory we shall mean a function having arbitrarily large zeros. The first theorem is an extension of a result of Utz [1] in which he considers $b(y) = y^{2n-1}$. We now prove a result for boundedness of solutions.

THEOREM I. If $a(t) > \alpha > 0$ and $da/dt \leq 0$ on $[T, \infty)$, $b(y)$ continuous, and $\lim_{y \rightarrow \pm\infty} B(y) = \int^{\infty} b(u) du = \infty$ then all solutions of

$$(1) \quad y'' + a(t)b(y) = 0$$

are bounded as $t \rightarrow \infty$.

PROOF. Multiply (1) by $2y'$ to get

$$(2) \quad 2y'y'' + 2a(t)b(y)y' = 0.$$

Upon integration by parts we have

$$(3) \quad (y'(u)^2)|_{t_0}^t + 2a(u)B(y(u))|_{t_0}^t - 2 \int_{t_0}^t \frac{da}{du} B(y) du = 0.$$

Thus

$$(4) \quad y'(t)^2 + 2a(t)B(y(t)) - 2 \int_{t_0}^t \frac{da}{du} B(y) du = K$$

where $K = y'(t_0)^2 + 2a(t_0)B(y(t_0))$. Now the above implies $|y'|$ and $|y|$ remain bounded. If not, the left side of (4) would become infinite which is impossible.

All solutions of (1) will be oscillatory if $\operatorname{sgn} y = \operatorname{sgn} b(y)$. To prove this we need the following lemma used by Utz [2].

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LEMMA. Suppose that $y(t)$ is a real function for which $y''(t)$ is defined for $t \geq T$.

- (i) If for all $t \geq T$, $y'(t) < 0$ and $y''(t) \leq 0$, then $\lim_{t \rightarrow \infty} y(t) = -\infty$.
- (ii) If for all $t \geq T$, $y'(t) > 0$, $y''(t) \geq 0$, then $\lim_{t \rightarrow \infty} y(t) = \infty$.

We now prove a result guaranteeing oscillation of solutions.

THEOREM II. If $a(t) \geq 0$, on $[T, \infty)$, $\int_T^\infty a(t) dt = \infty$, $yb(y) > 0$ for $y \neq 0$, $b(y)$ continuous, then any bounded solutions of $y'' + a(t)b(y) = 0$ is necessarily oscillatory.

PROOF. Suppose $y(t)$ did not oscillate. Then for large t , y is of fixed sign. We show y must be monotonic. Assume $y > 0$ (a similar argument works for $y < 0$). If $y'(t) = 0$, then $y'' = -a(t)b(y) < 0$. This implies $y'(t)$ cannot vanish for arbitrarily large t since $y(t)$ would then have infinitely many relative maxima which is impossible. Therefore, $y'(t)$ is of fixed sign for large t . Now $y'(t)$ must be positive. If $y'(t) < 0$, then $\lim_{t \rightarrow \infty} y(t) = -\infty$ by applying the above lemma. But this contradicts the boundedness of $y(t)$. In fact we must have $\lim_{t \rightarrow \infty} y'(t) = 0$ and $\lim_{t \rightarrow \infty} y(t) = c > 0$. Now by integrating (1) we get

$$(5) \quad y'(t) = y(T) - \int_T^t a(u)b(y(u)) du.$$

But

$$\lim_{t \rightarrow \infty} \int_T^t a(u)b(y(u)) du = \infty$$

since $\int_T^\infty a(u) du = \infty$ and $\lim_{u \rightarrow \infty} b(y(u)) = b(c) > 0$. So $y'(t) < 0$ for t sufficiently large, a contradiction. Thus, $y(t)$ must oscillate.

REFERENCES

1. W. R. Utz, *Properties of solutions of $u'' + g(t)u^{2n-1} = 0$* , Monatsh. Math. **66** (1962), 55–60. MR **25** #2275.
2. ———, *A note on second order nonlinear differential equations*, Proc. Amer. Math. Soc. **7** (1956), 1047–1048. MR **19**, 141.

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