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SOLUTIONS OF SOME PERIODIC STIELTJES INTEGRAL EQUATIONS

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ABSTRACT. Nonlinear periodic perturbations of a family of linear periodic Stieltjes integral equations are considered and sufficient conditions are given for the existence of a periodic solution for one member of the family. Conditions are given under which the solutions of the family approach the periodic solution asymptotically. A Floquet type theorem for periodic Stieltjes integral equations and several examples are given.

Let S be an interval of numbers of the form $[\tau, \infty)$, let $\{G, |\cdot|\}$ be a complete normed abelian group and denote the set of all functions from G to G by H. A function V from $S\times S$ into H such that $V(x, y)+V(y, z)=V(x, z)$ whenever y is between x and z will be called an order additive function. Suppose that each of V_1 and V_2 is an order additive function such that

(1) $V_1(x, y)0=0$ for all (x, y) in $S\times S$ where 0 is the zero element of G;

(2) there are order additive functions α_1, α_2 from $S\times S$ to the nonnegative numbers such that

$$|V_i(x, y)p - V_i(x, y)q| \leqq \alpha_i(x, y)\,|p-q|, \qquad |V_2(x, y)0| \leqq \alpha_2(x, y),$$

for each (x, y) in $S\times S$, (p, q) in $G\times G$ and $i=1, 2$. We will also suppose that $V_2(x, y)0\neq 0$ for some (x, y) in $S\times S$.

The first result in this paper is a Floquet theorem which says that under periodicity conditions on V_1, a knowledge of the solution of

$$h(t) = p + (R)\int_t^\tau V_1 h \tag{1}$$

over a period $\tau\leqq t\leqq\tau+\omega$ gives the values for all $\tau\leqq t$. The second theorem describes conditions under which there is a p in G such that the solution of

$$h(t) = p + (R)\int_t^\tau (V_1 + V_2)h \tag{2}$$

is periodic. Mac Nerney gives the existence and uniqueness theorems for

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equations (1) and (2) in [3]. The second result generalizes a well-known theorem concerning the existence of a periodic solution of a perturbation of a noncritical system of linear differential equations. Here a variation of parameter theorem obtained by Reneke [6] is the primary tool used. Theorem 3 gives conditions under which the solution obtained from Theorem 2 is stable.

Let K map $S\times S$ into H and have the property that there is a positive number ω such that $K(x, y)=K(x+\omega, y+\omega)$ for all (x, y) in $S\times S$; then we will say that K has the ω period property.

REMARK. If K is order additive it is easy to see that K has the ω period property if and only if ω is a positive number such that

$$K(x, x+\omega) = K(y, y+\omega) \quad \text{and} \quad K(x+\omega, x) = K(y+\omega, y).$$

Mac Nerney [3] shows that there is a function W_1 from $S\times S$ into H defined by

$$W_1(x, y)p = {}_x\prod^y (I + V_1)p \tag{3}$$

for (x, y) in $S\times S$ and p in G, that W_1 is order multiplicative (i.e. $W_1(x, y)W_1(y, z)= W_1(x, z)$ for all y between x and z), and that h given by $h(t)= W_1(t, \tau)p$ is the unique solution of (1). Here I is the identity map on G.

LEMMA 1. *If V_1 has the ω period property then W_1 has the ω period property.*

The proof follows from the definition of W_1 and a simple translation of partitions of the interval x to y into the interval $x+\omega$ to $y+\omega$.

THEOREM 1. *If V_1 has the ω period property, n is a positive integer and $t\geqq\tau$ then*

$$W_1(t + n\omega, \tau) = W_1(t, \tau)W_1(\tau + \omega, \tau)^n.$$

The proof follows by induction using the order multiplicative property of W_1.

REMARK 1. Under the hypothesis of Theorem 1, if $W_1(\tau+t, \tau)$ is known for $0\leqq t\leqq\omega$ then the solution h of (1) is known for all $t\geqq\tau$.

REMARK 2. If $W(\tau+\omega, \tau)=I$ then for any p in G the solution of (1) is periodic (i.e. $h(t+\omega)=h(t)$ for $t\geqq\tau$).

REMARK 3. Let A be a continuous $n\times n$ matrix valued function from S with period ω, let $V_1(x, y)p=-[\int_x^y A(s)\, ds]p$ for p in R^n and let h_0 be a continuous function from $[\tau, \infty)$ into R^n, then

$$(R)\int_t^\tau Vh_0 = \int_\tau^t A(s)h_0(s)\, ds.$$

Consequently, since the sequence $h_n(t)$ given by $h_n(t)=p+(R)\int_t^\tau Vh_{n-1}$, $n=1, 2, \cdots$, converges uniformly to the solution of (1) (see Theorem 2 in [3]) we have $W_1(t, \tau)p=X(t)X^{-1}(\tau)p$ where X is a fundamental matrix of solutions of $x'=A(t)x$. Theorem 1 in this case is the usual Floquet theorem (see Hale [1]).

REMARK 4. Let $S=[0,\infty)$, let A be an $n\times n$ matrix valued function from S which has local bounded variation and let $(\ ):S\rightarrow R$ be given by $(t)=n-1$ for $n-1<t\leqq n$ where n is a positive integer. The solution of the vector difference equation $y(n+1)=A(n)y(n)$ satisfies

$$y(n) = y(0) + (R)\int_n^0 Vy,$$

where $V(u, v)y=-(R)\int_u^v (A-I^*)\,d(\ )\cdot y$ and conversely. Here I^* is the $n\times n$ identity matrix. Consequently, Theorem 1 is a Floquet theorem for difference equations.

REMARK 5. If $V_2(x, y)0\neq 0$ for all (x, y) in $S\times S$ we can establish similar results for an integral equation of the form

$$h(t) = p + (R)\int_t^\tau V_2 h. \tag{4}$$

Following Mac Nerney [3], let $G^*=\{(g, m):g\in G, m\in\{0, 1\}\}$, let addition in G^* be componentwise and addition in the second component be modulo 2 and let $|(g, m)|=|g|+m$. It is easy to verify that $(G^*, |\cdot|)$ is a complete normed abelian group. Let V^* be a map from $S\times S$ to H^* given by

$$V^*(x, y)(g, m) = (V_2(x, y)p - V_2(x, y)0 + mV_2(x, y)0, 0).$$

We note that $V^*(x, y)0 = 0$ and that if V_2 has the ω period property so does V^*. It is easy to see that for $W^*:S\times S\rightarrow H^*$ given by

$$W^*(x, y)(g, m) = {}_x\prod^y(I + V^*)(g, m)$$

and $W_3:S\times S\rightarrow H$ given by

$$W^*(x, y)(g, 1) = (W_3(x, y)(g, 1), 1)$$

the function $h=W_3(\ , \tau)(p, 1)$ is the solution of (4). Consequently if $W^*(\tau+t, \tau)$ is known for $0\leqq t\leqq\omega$ the solution of (4) is known for all $t\geqq\tau$.

For the remaining portion of the paper let $V_1(x, y)$ be a group homomorphism of G for each (x, y) in $S\times S$. It is easy to see that W_1 defined by (3) also has its values in the group homomorphisms of G. Let ω be a positive number and QC be the collection of quasi-continuous functions h

from $[\tau, \tau+\omega]$ to G (i.e., $h(t^+)$ exists for $\tau \leqq t < \tau+\omega$ in S, $h(t^-)$ exists for $\tau < t \leqq \tau+\omega$) with $h(\tau)=h(\tau+\omega)$, and for h in QC let

$$\|h\| = \sup_{\tau \leqq t \leqq \tau+\omega} |h(t)|.$$

LEMMA 2. *For h in QC the function k given by*

$$k(t) = (L, R)\int_t^\tau W_1(t, \ )V_2 h, \qquad \tau \leqq t \leqq \tau + \omega,$$

exists and has bounded variation.

The proof is similar to the proof of Lemma 2.2 in Mac Nerney [3] and thus will not be given here.

Mac Nerney [3] also shows that the function $\mu_1 : S \times S$ to the numbers not less than 1 given by

$$\mu_1(x, y) = {}_x\prod^y (1 + \alpha_1)$$

exists and $|W_1(x, y)p| \leqq \mu_1(x, y)|p|$.

THEOREM 2. *If V_1, V_2 have the ω period property, if the map $I - W_1(\omega+\tau, \tau)$ is reversible and there is an $M > 0$ such that*

(a) $$|[I - W_1(\omega + \tau, \tau)]^{-1} g| \leqq M\,|g| \quad \textit{for } g \in G,$$

and if

(b) $$(L)\int_{\tau+\omega}^{\tau} \mu_1(\tau + \omega, \ )\alpha_2 < \frac{1}{\mu_1(\tau + \omega, \tau)M + 1},$$

then there is a unique p in G such that the solution h of (2) *has period ω (i.e. $h(t+\omega)=h(t)$ for $t \geqq \tau$).*

PROOF. For f in QC define

$$P_f = [I - W_1(\tau + \omega, \tau)]^{-1}(L, R)\int_{\tau+\omega}^{\tau} W_1(\tau + \omega, \ )V_2 f,$$

and

$$Tf(t) = W_1(t, \tau)P_f + (L, R)\int_t^\tau W_1(t, \ )V_2 f.$$

We note that since

$$\begin{aligned} &Tf(\tau) - Tf(\tau + \omega) \\ &\quad = [I - W_1(\tau + \omega, \tau)]P_f - (L, R)\int_{\tau+\omega}^{\tau} W_1(\tau + \omega, \ )V_2 f = 0, \end{aligned}$$

T maps QC into itself ($W_1(\ , \tau)$ has bounded variation, see Mac Nerney

[3]). Also for f, h in QC we have

$$|P_f - P_h| \leqq M \left|(L, R)\int_{\tau+\omega}^{\tau} W_1(\tau + \omega, \ )[V_2 f - V_2 h]\right|,$$
$$\leqq M(L)\int_{\tau+\omega}^{\tau} \mu_1(\tau + \omega, \ )\alpha_2 \, \|f - h\|;$$

and similarly

$$|Tf(t) - Th(t)| \leqq \|f - h\| \, [M\mu_1(\tau + \omega, \tau) + 1](L)\int_{\tau+\omega}^{\tau} \mu_1(\tau + \omega, \ )\alpha_2.$$

Consequently by the Contraction Mapping Theorem there is an f in QC such that

$$(5) \qquad f(t) = W_1(t, \tau)P_f + (L, R)\int_t^{\tau} W_1(t, \ )V_2 f, \qquad \tau \leqq t \leqq \tau + \omega.$$

Let h be the function defined on $\tau \leqq t$ into G by $h(t) = f(t - n\omega)$ where n is the positive integer such that $\tau + n\omega \leqq t < (n+1)\omega + \tau$. Clearly h is quasi-continuous and periodic and

$$h(t) = W_1(t - n\omega, \tau)P_f + (L, R)\int_{t-n\omega}^{\tau} W_1(t - n\omega, \ )V_2 f$$
$$= W_1(t, \tau + n\omega)P_f + (L, R)\int_t^{\tau+n\omega} W_1(t, \ )V_2 h$$
$$= W_1(t, \tau + n\omega)\left[W_1(\tau + n\omega, \tau)P_f + (L, R)\int_{\tau+n\omega}^{\tau} W_1(n\omega + \tau, \ )V_2 h\right]$$
$$+ (L, R)\int_t^{\tau+n\omega} W_1(t, \ )V_2 h;$$

thus

$$(6) \qquad h(t) = W_1(t, \tau)P_f + (L, R)\int_t^{\tau} W_1(t, \ )V_2 h, \qquad \tau \leqq t.$$

Reneke [6] has shown that h is a solution of (6) if and only if h is a solution of (2) with p replaced by P_f.

For the uniqueness part of the theorem we note that if h^* is a solution of (2) with period ω then by Reneke's result $Th^* = h^*$. Since T has exactly one fixed point $h = h^*$ and thus $p = P_f$.

REMARK 6. Take V_1 as in Remark 3 and let V_2 be given by

$$V_2(x, y)p = -\int_x^y f(s, p)\, ds, \qquad (x, y) \text{ in } S \times S, \qquad p \text{ in } G,$$

where f is a continuous function from $S \times R^n$ into R^n with $f(t+\omega, p) = f(t, p)$ and $|f(t, p) - f(t, q)| \leqq L\,|p - q|$. The solution of (2) satisfies

$$x' = A(t)x + f(t, x);$$

thus Theorem 2 generalizes a well-known theorem in ordinary differential equations (see Hale [**1**, Theorem 5-1]).

For any f in H satisfying a Lipschitz condition and $f(0)=0$ let

$$N(f) = \sup\{|f(p)|/|p| : p \in G, p \neq 0\}.$$

In the case G is a Banach space Martin has shown [**5**] that the functions σ and π on $S \times S$ given by

$$\sigma(a, b) = {}_a\sum^b [N(I + V_1) - 1], \qquad \pi(a, b) = {}_a\prod^b N(W_1)$$

exists, $|W_1(t, \tau)p| \leqq \pi(t, \tau)|p|$, σ is order additive and for any real number δ, the unique solution of the integral equation $\chi(t)=\delta+(R)\int_t^\tau \sigma\chi$ is given by $\chi(t)=\pi(t, \tau)\delta$.

REMARK 7. In the case that G is a Banach space, condition (b) in Theorem 2 may be replaced by

$$\sup_{\tau \leqq t \leqq \tau+\omega} \left\{ \pi(t, \tau) M(L) \int_{\tau+\omega}^{\tau} \pi(\tau + \omega, \ )\alpha_2 + (L)\int_t^\tau \pi(t, \ )\alpha_2 \right\} < 1$$

and the conclusion of the theorem follows using the same proof.

THEOREM 3. *Let G be a Banach space and let the hypothesis of Theorem 2 hold. Further suppose that for any number δ, the solutions k_δ of*

$$k(t) = \delta + (R)\int_t^\tau (\sigma + \alpha_2)k \tag{7}$$

satisfy $\lim_{t\to\infty} k_\delta(t)=0$ *where σ is defined above. If h is the periodic solution of Theorem* 2 *and h^* is the solution of* (2) *with $h^*(\tau)=p^*$,*

$$\lim_{t\to\infty}[h^*(t) - h(t)] = 0.$$

PROOF. According to the variation of parameters formula devised by Reneke [**6**],

$$h^*(t) = W_1(t, \tau)p^* + (L, R)\int_t^\tau W_1(t, \ )V_2 h^*$$

and h satisfies a similar equation. Thus for $\lambda(t)=|h^*(t)-h(t)|$ we have

$$\lambda(t) \leqq \pi(t, \tau)\,|p - p^*| + (L, R)\int_t^\tau \pi(t, \ )\alpha_2\lambda.$$

The function k_δ where $\delta=|p-p^*|$ satisfies

$$k_\delta(t) = \pi(t, \tau)\,|p - p^*| + (L, R)\int_t^\tau \pi(t, \ )\alpha_2 k_\delta$$

and by a Gronwall type result given by Marrah [**4**] we have $\lambda(t) \leqq k_\delta(t)$.

REMARK 8. For the V_1, V_2 specified in Remark 6,

$$\sigma(a, b) = \int_a^b \mu[A(s)]\, ds \qquad \text{if } a \leqq b,$$

$$= -\int_a^b \mu[A(s)]\, ds \quad \text{if } a > b,$$

where μ is the logarithmic norm

$$\mu[A(s)] = \lim_{h\to 0^+} (|I + hA(s)| - 1)/h;$$

consequently, the solution of equation (7) satisfies

$$dk/dt = (\mu[A(t)] + L)k.$$

In this case Theorem 3 is a restatement of a known result that if $\mu[A(s)]+L<0$ then the periodic solution furnished by Theorem 2 is asymptotically stable.

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