

SOLUTIONS OF SOME PERIODIC STIELTJES INTEGRAL EQUATIONS

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ABSTRACT. Nonlinear periodic perturbations of a family of linear periodic Stieltjes integral equations are considered and sufficient conditions are given for the existence of a periodic solution for one member of the family. Conditions are given under which the solutions of the family approach the periodic solution asymptotically. A Floquet type theorem for periodic Stieltjes integral equations and several examples are given.

Let S be an interval of numbers of the form $[\tau, \infty)$, let $\{G, |\cdot|\}$ be a complete normed abelian group and denote the set of all functions from G to G by H . A function V from $S \times S$ into H such that $V(x, y) + V(y, z) = V(x, z)$ whenever y is between x and z will be called an order additive function. Suppose that each of V_1 and V_2 is an order additive function such that

- (1) $V_1(x, y)0 = 0$ for all (x, y) in $S \times S$ where 0 is the zero element of G ;
- (2) there are order additive functions α_1, α_2 from $S \times S$ to the non-negative numbers such that

$$|V_i(x, y)p - V_i(x, y)q| \leq \alpha_i(x, y) |p - q|, \quad |V_2(x, y)0| \leq \alpha_2(x, y),$$

for each (x, y) in $S \times S$, (p, q) in $G \times G$ and $i = 1, 2$. We will also suppose that $V_2(x, y)0 \neq 0$ for some (x, y) in $S \times S$.

The first result in this paper is a Floquet theorem which says that under periodicity conditions on V_1 , a knowledge of the solution of

$$(1) \quad h(t) = p + (R) \int_t^{\tau} V_1 h$$

over a period $\tau \leq t \leq \tau + \omega$ gives the values for all $\tau \leq t$. The second theorem describes conditions under which there is a p in G such that the solution of

$$(2) \quad h(t) = p + (R) \int_t^{\tau} (V_1 + V_2)h$$

is periodic. Mac Nerney gives the existence and uniqueness theorems for

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equations (1) and (2) in [3]. The second result generalizes a well-known theorem concerning the existence of a periodic solution of a perturbation of a noncritical system of linear differential equations. Here a variation of parameter theorem obtained by Reneke [6] is the primary tool used. Theorem 3 gives conditions under which the solution obtained from Theorem 2 is stable.

Let K map $S \times S$ into H and have the property that there is a positive number ω such that $K(x, y) = K(x + \omega, y + \omega)$ for all (x, y) in $S \times S$; then we will say that K has the ω period property.

REMARK. If K is order additive it is easy to see that K has the ω period property if and only if ω is a positive number such that

$$K(x, x + \omega) = K(y, y + \omega) \quad \text{and} \quad K(x + \omega, x) = K(y + \omega, y).$$

Mac Nerney [3] shows that there is a function W_1 from $S \times S$ into H defined by

$$(3) \quad W_1(x, y)p = {}_x \prod_1^y (I + V_1)p$$

for (x, y) in $S \times S$ and p in G , that W_1 is order multiplicative (i.e. $W_1(x, y)W_1(y, z) = W_1(x, z)$ for all y between x and z), and that h given by $h(t) = W_1(t, \tau)p$ is the unique solution of (1). Here I is the identity map on G .

LEMMA 1. *If V_1 has the ω period property then W_1 has the ω period property.*

The proof follows from the definition of W_1 and a simple translation of partitions of the interval x to y into the interval $x + \omega$ to $y + \omega$.

THEOREM 1. *If V_1 has the ω period property, n is a positive integer and $t \geq \tau$ then*

$$W_1(t + n\omega, \tau) = W_1(t, \tau)W_1(\tau + \omega, \tau)^n.$$

The proof follows by induction using the order multiplicative property of W_1 .

REMARK 1. Under the hypothesis of Theorem 1, if $W_1(\tau + t, \tau)$ is known for $0 \leq t \leq \omega$ then the solution h of (1) is known for all $t \geq \tau$.

REMARK 2. If $W(\tau + \omega, \tau) = I$ then for any p in G the solution of (1) is periodic (i.e. $h(t + \omega) = h(t)$ for $t \geq \tau$).

REMARK 3. Let A be a continuous $n \times n$ matrix valued function from S with period ω , let $V_1(x, y)p = -[\int_x^y A(s) ds]p$ for p in R^n and let h_0 be a continuous function from $[\tau, \infty)$ into R^n , then

$$(R) \quad \int_t^\tau V h_0 = \int_\tau^t A(s) h_0(s) ds.$$

Consequently, since the sequence $h_n(t)$ given by $h_n(t) = p + (R) \int_t^r V h_{n-1}$, $n = 1, 2, \dots$, converges uniformly to the solution of (1) (see Theorem 2 in [3]) we have $W_1(t, \tau)p = X(t)X^{-1}(\tau)p$ where X is a fundamental matrix of solutions of $x' = A(t)x$. Theorem 1 in this case is the usual Floquet theorem (see Hale [1]).

REMARK 4. Let $S = [0, \infty)$, let A be an $n \times n$ matrix valued function from S which has local bounded variation and let $(\cdot) : S \rightarrow R$ be given by $(t) = n - 1$ for $n - 1 < t \leq n$ where n is a positive integer. The solution of the vector difference equation $y(n + 1) = A(n)y(n)$ satisfies

$$y(n) = y(0) + (R) \int_n^0 V y,$$

where $V(u, v)y = -(R) \int_u^v (A - I^*) d(\cdot) \cdot y$ and conversely. Here I^* is the $n \times n$ identity matrix. Consequently, Theorem 1 is a Floquet theorem for difference equations.

REMARK 5. If $V_2(x, y)0 \neq 0$ for all (x, y) in $S \times S$ we can establish similar results for an integral equation of the form

$$(4) \quad h(t) = p + (R) \int_t^r V_2 h.$$

Following Mac Nerney [3], let $G^* = \{(g, m) : g \in G, m \in \{0, 1\}\}$, let addition in G^* be componentwise and addition in the second component be modulo 2 and let $|(g, m)| = |g| + m$. It is easy to verify that $(G^*, |\cdot|)$ is a complete normed abelian group. Let V^* be a map from $S \times S$ to H^* given by

$$V^*(x, y)(g, m) = (V_2(x, y)p - V_2(x, y)0 + mV_2(x, y)0, 0).$$

We note that $V^*(x, y)0 = 0$ and that if V_2 has the ω period property so does V^* . It is easy to see that for $W^* : S \times S \rightarrow H^*$ given by

$$W^*(x, y)(g, m) = {}_x \prod_1^y (I + V^*)(g, m)$$

and $W_3 : S \times S \rightarrow H$ given by

$$W^*(x, y)(g, 1) = (W_3(x, y)(g, 1), 1)$$

the function $h = W_3(\cdot, \tau)(p, 1)$ is the solution of (4). Consequently if $W^*(\tau + t, \tau)$ is known for $0 \leq t \leq \omega$ the solution of (4) is known for all $t \geq \tau$.

For the remaining portion of the paper let $V_1(x, y)$ be a group homomorphism of G for each (x, y) in $S \times S$. It is easy to see that W_1 defined by (3) also has its values in the group homomorphisms of G . Let ω be a positive number and QC be the collection of quasi-continuous functions h

from $[\tau, \tau + \omega]$ to G (i.e., $h(t^+)$ exists for $\tau \leq t < \tau + \omega$ in S , $h(t^-)$ exists for $\tau < t \leq \tau + \omega$) with $h(\tau) = h(\tau + \omega)$, and for h in QC let

$$\|h\| = \sup_{\tau \leq t \leq \tau + \omega} |h(t)|.$$

LEMMA 2. For h in QC the function k given by

$$k(t) = (L, R) \int_t^\tau W_1(t, \cdot) V_2 h, \quad \tau \leq t \leq \tau + \omega,$$

exists and has bounded variation.

The proof is similar to the proof of Lemma 2.2 in Mac Nerney [3] and thus will not be given here.

Mac Nerney [3] also shows that the function $\mu_1: S \times S$ to the numbers not less than 1 given by

$$\mu_1(x, y) = {}_x \prod_1^y (1 + \alpha_1)$$

exists and $|W_1(x, y)p| \leq \mu_1(x, y)|p|$.

THEOREM 2. If V_1, V_2 have the ω period property, if the map $I - W_1(\omega + \tau, \tau)$ is reversible and there is an $M > 0$ such that

(a) $|[I - W_1(\omega + \tau, \tau)]^{-1}g| \leq M|g|$ for $g \in G$,

and if

(b) $(L) \int_{\tau+\omega}^\tau \mu_1(\tau + \omega, \cdot) \alpha_2 < \frac{1}{\mu_1(\tau + \omega, \tau)M + 1}$,

then there is a unique p in G such that the solution h of (2) has period ω (i.e. $h(t + \omega) = h(t)$ for $t \geq \tau$).

PROOF. For f in QC define

$$P_f = [I - W_1(\tau + \omega, \tau)]^{-1}(L, R) \int_{\tau+\omega}^\tau W_1(\tau + \omega, \cdot) V_2 f,$$

and

$$Tf(t) = W_1(t, \tau)P_f + (L, R) \int_t^\tau W_1(t, \cdot) V_2 f.$$

We note that since

$$\begin{aligned} Tf(\tau) - Tf(\tau + \omega) &= [I - W_1(\tau + \omega, \tau)]P_f - (L, R) \int_{\tau+\omega}^\tau W_1(\tau + \omega, \cdot) V_2 f = 0, \end{aligned}$$

T maps QC into itself ($W_1(\cdot, \tau)$ has bounded variation, see Mac Nerney

[3]). Also for f, h in QC we have

$$\begin{aligned} |P_f - P_h| &\leq M \left| (L, R) \int_{\tau+\omega}^{\tau} W_1(\tau + \omega, \cdot) [V_2 f - V_2 h] \right|, \\ &\leq M(L) \int_{\tau+\omega}^{\tau} \mu_1(\tau + \omega, \cdot) \alpha_2 \|f - h\|; \end{aligned}$$

and similarly

$$|Tf(t) - Th(t)| \leq \|f - h\| [M\mu_1(\tau + \omega, \tau) + 1] (L) \int_{\tau+\omega}^{\tau} \mu_1(\tau + \omega, \cdot) \alpha_2.$$

Consequently by the Contraction Mapping Theorem there is an f in QC such that

$$(5) \quad f(t) = W_1(t, \tau)P_f + (L, R) \int_t^{\tau} W_1(t, \cdot) V_2 f, \quad \tau \leq t \leq \tau + \omega.$$

Let h be the function defined on $\tau \leq t$ into G by $h(t) = f(t - n\omega)$ where n is the positive integer such that $\tau + n\omega \leq t < (n+1)\omega + \tau$. Clearly h is quasi-continuous and periodic and

$$\begin{aligned} h(t) &= W_1(t - n\omega, \tau)P_f + (L, R) \int_{t-n\omega}^{\tau} W_1(t - n\omega, \cdot) V_2 f \\ &= W_1(t, \tau + n\omega)P_f + (L, R) \int_t^{\tau+n\omega} W_1(t, \cdot) V_2 h \\ &= W_1(t, \tau + n\omega) \left[W_1(\tau + n\omega, \tau)P_f + (L, R) \int_{\tau+n\omega}^{\tau} W_1(n\omega + \tau, \cdot) V_2 h \right] \\ &\quad + (L, R) \int_t^{\tau+n\omega} W_1(t, \cdot) V_2 h; \end{aligned}$$

thus

$$(6) \quad h(t) = W_1(t, \tau)P_f + (L, R) \int_t^{\tau} \dot{W}_1(t, \cdot) V_2 h, \quad \tau \leq t.$$

Reneke [6] has shown that h is a solution of (6) if and only if h is a solution of (2) with p replaced by P_f .

For the uniqueness part of the theorem we note that if h^* is a solution of (2) with period ω then by Reneke's result $Th^* = h^*$. Since T has exactly one fixed point $h = h^*$ and thus $p = P_f$.

REMARK 6. Take V_1 as in Remark 3 and let V_2 be given by

$$V_2(x, y)p = - \int_x^y f(s, p) ds, \quad (x, y) \text{ in } S \times S, \quad p \text{ in } G,$$

where f is a continuous function from $S \times R^n$ into R^n with $f(t + \omega, p) = f(t, p)$ and $|f(t, p) - f(t, q)| \leq L|p - q|$. The solution of (2) satisfies

$$x' = A(t)x + f(t, x);$$

thus Theorem 2 generalizes a well-known theorem in ordinary differential equations (see Hale [1, Theorem 5-1]).

For any f in H satisfying a Lipschitz condition and $f(0)=0$ let

$$N(f) = \sup\{|f(p)|/|p| : p \in G, p \neq 0\}.$$

In the case G is a Banach space Martin has shown [5] that the functions σ and π on $S \times S$ given by

$$\sigma(a, b) = {}_a\sum^b [N(I + V_1) - 1], \quad \pi(a, b) = {}_a\prod^b N(W_1)$$

exists, $|W_1(t, \tau)p| \leq \pi(t, \tau)|p|$, σ is order additive and for any real number δ , the unique solution of the integral equation $\chi(t) = \delta + (R) \int_t^\tau \sigma \chi$ is given by $\chi(t) = \pi(t, \tau)\delta$.

REMARK 7. In the case that G is a Banach space, condition (b) in Theorem 2 may be replaced by

$$\sup_{\tau \leq t \leq \tau + \omega} \left\{ \pi(t, \tau)M(L) \int_{\tau + \omega}^\tau \pi(\tau + \omega, \)\alpha_2 + (L) \int_t^\tau \pi(t, \)\alpha_2 \right\} < 1$$

and the conclusion of the theorem follows using the same proof.

THEOREM 3. Let G be a Banach space and let the hypothesis of Theorem 2 hold. Further suppose that for any number δ , the solutions k_δ of

$$(7) \quad k(t) = \delta + (R) \int_t^\tau (\sigma + \alpha_2)k$$

satisfy $\lim_{t \rightarrow \infty} k_\delta(t) = 0$ where σ is defined above. If h is the periodic solution of Theorem 2 and h^* is the solution of (2) with $h^*(\tau) = p^*$,

$$\lim_{t \rightarrow \infty} [h^*(t) - h(t)] = 0.$$

PROOF. According to the variation of parameters formula devised by Reneke [6],

$$h^*(t) = W_1(t, \tau)p^* + (L, R) \int_t^\tau W_1(t, \)V_2h^*$$

and h satisfies a similar equation. Thus for $\lambda(t) = |h^*(t) - h(t)|$ we have

$$\lambda(t) \leq \pi(t, \tau) |p - p^*| + (L, R) \int_t^\tau \pi(t, \)\alpha_2\lambda.$$

The function k_δ where $\delta = |p - p^*|$ satisfies

$$k_\delta(t) = \pi(t, \tau) |p - p^*| + (L, R) \int_t^\tau \pi(t, \)\alpha_2k_\delta$$

and by a Gronwall type result given by Marrah [4] we have $\lambda(t) \leq k_\delta(t)$.

REMARK 8. For the V_1, V_2 specified in Remark 6,

$$\begin{aligned}\sigma(a, b) &= \int_a^b \mu[A(s)] ds && \text{if } a \leq b, \\ &= - \int_a^b \mu[A(s)] ds && \text{if } a > b,\end{aligned}$$

where μ is the logarithmic norm

$$\mu[A(s)] = \lim_{h \rightarrow 0^+} (|I + hA(s)| - 1)/h;$$

consequently, the solution of equation (7) satisfies

$$dk/dt = (\mu[A(t)] + L)k.$$

In this case Theorem 3 is a restatement of a known result that if $\mu[A(s)] + L < 0$ then the periodic solution furnished by Theorem 2 is asymptotically stable.

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