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REFLEXIVITY OF $L(E, F)$

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ABSTRACT. Let E and F be two Banach spaces both having the approximation property. The space $L(E, F)$ is reflexive if and only if (a) both E and F are reflexive, (b) every continuous linear operator from E into F is compact. Thus $L(l^p, l^q)$ is reflexive for $1<q<p<\infty$.

In this note we piece together some results of Grothendieck to ascertain when $L(E, F)$, the space of continuous linear operators from a Banach space E to a Banach space F, is reflexive. This condition sometimes holds when both E and F have infinite dimension. This demolishes an apparently popular supposition that $L(E, F)$ is reflexive only if one space is reflexive and the other finite dimensional.

The space of compact linear operators from E to F is denoted by $C(E, F)$. It is known that $C(E, F)$ is a closed subspace of $L(E, F)$. A Banach space E is said to have the *approximation property* if for each Banach space F, $C(E, F)$ is the closure in $L(E, F)$ of those operators of finite rank; in other words if $C(E, F)$ is the closed linear span of the set of all one dimensional operators $x' \otimes y$. The value of $x' \otimes y$ with $x' \in E'$, the dual space of E, and $y \in F$ is given by the formula

$$x' \otimes y(x) = \langle x, x' \rangle y, \qquad x \in E.$$

Equivalent formulations of the approximation property can be found on pp. 164–165 of [**1**].

THEOREM. *Let E and F be two Banach spaces both having the approximation property. The space $L(E, F)$ is reflexive if and only if the following pair of conditions holds*:

(a) *both E and F are reflexive*,

(b) $L(E, F) = C(E, F)$.

PROOF. Suppose ϕ is in $C(E, F)'$. For x' in E' the equation

$$\langle T_\phi x', y \rangle = \langle x' \otimes y, \phi \rangle, \qquad y \in F,$$

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determines a linear functional $T_\phi x'$ on F. Since $|\langle T_\phi x', y\rangle| \leqq \|x' \oplus y\| \, \|\phi\| = \|x'\| \, \|y\| \, \|\phi\|$, $T_\phi x'$ is in F' and $T_\phi \in L(E', F')$ with $\|T_\phi\| \leqq \|\phi\|$. If $T_\phi = T_\theta$ then

$$\langle x' \otimes y, \phi\rangle = \langle x' \otimes y, \theta\rangle$$

for each x' in E' and y in F. Since E has the approximation property we have $\phi = \theta$. The space of all T_ϕ for ϕ in $C(E, F)'$ is called the space of *integral operators* and is denoted by $I(E, F)$. Let $I(E, F)$ be given the topology of identification with $C(E, F)'$.

For x'' in E'' and y' in F', $x'' \otimes y'$ is in $I(E', F')$ determined by ϕ in $C(E, F)$ for which

$$\langle T, \phi\rangle = \langle T'y', x''\rangle, \qquad T \in C(E, F).$$

Here T' denotes the adjoint of T. The norm of this ϕ is $\|x''\| \, \|y'\|$; thus $\|x'' \otimes y'\|_I = \|x''\| \, \|y'\|$.

Since $I(E', F')$ is complete it must contain all operators of the form

$$S = \sum_{n=1}^{\infty} x_n'' \otimes y_n' \tag{1}$$

where $\sum_{n=1}^{\infty} \|x_n''\| \, \|y_n'\| < \infty$. The expansion for S given by (1) is not unique. An operator having such an expansion is called *nuclear*; the set of all nuclear operators in $I(E', F')$ is denoted by $N(E', F')$. Clearly, $N(E', F')$ is contained in the closed linear span in $I(E', F')$ of all one dimensional operators $x'' \otimes y'$.

NECESSITY OF (a). Let y_0 be a vector in F with $\|y_0\| = 1$. The correspondence of x' in E' to $x' \otimes y_0$ in $L(E, F)$ is an isometry for E' onto a subspace of $L(E, F)$. Thus if $L(E, F)$ is reflexive so is E' and hence E. A similar proof shows F reflexive.

NECESSITY OF (b). Every ϕ in $C(E, F)''$ can be considered as a continuous linear functional on $I(E', F')$. For each s'' in E'' we determine $T_\phi x''$ in F'' by the rule

$$\langle y', T_\phi x''\rangle = \langle x'' \otimes y', \phi\rangle, \qquad y' \in F'.$$

Since

$$|\langle x'' \otimes y', \phi\rangle| \leqq \|x''\| \, \|y'\| \, \|\phi\|,$$

it follows that T_ϕ is continuous and $\|T_\phi\| \leqq \|\phi\|$. The correspondence of ϕ in $C(E, F)''$ to T_ϕ in $L(E'', F'')$ is linear and continuous. In order to show it is one-to-one we appeal to a result of Grothendieck [**1**, p. 134] which asserts that if F' is reflexive then $I(E', F') = N(E', F')$ and

$$\|S\|_I = \inf\left\{\sum_{n=1}^{\infty} \|x_n''\| \, \|y_n\| : S \text{ has the form } (1)\right\}.$$

If $T_\phi=0$ we then have

$$\langle y', T_\phi x''\rangle = \langle x'' \otimes y', \phi\rangle = 0$$

for each one-dimensional operator $x''\otimes y'$, from which we conclude $\phi=0$.

The correspondence from ϕ in $C(E, F)''$ to T_ϕ in $L(E'', F'')$ is also onto when F' is reflexive. For T in $L(E'', F'')$ define ϕ on $N(E', F')$ by

$$\left\langle \sum_{n=1}^{\infty} x_n'' \otimes y_n', \phi \right\rangle = \sum_{n=1}^{\infty} \langle Tx_n'', y_n'\rangle.$$

Since $F''=F$ has the approximation property, the value of ϕ does not depend on the representation of $\sum_{n=1}^{\infty} x_n''\otimes y_n'$ [**1**, pp. 164–165]. We then have

$$\sum_{n=1}^{\infty} |\langle Tx_n'', y_n'\rangle| \leqq \|T\| \sum_{n=1}^{\infty} \|x_n''\| \, \|y_n'\|,$$

so that for each S in $N(E', F')$,

$$|\langle S, \phi\rangle| \leqq \|S\|_I \, \|T\|.$$

Therefore ϕ is continuous on $I(E', F')$ and $T=T_\phi$.

For each T in $L(E'', F'')=L(E, F)$ there is ϕ in $C(E, F)''$ such that $\langle x'\otimes y', \phi\rangle=\langle Tx, y'\rangle$ for each $x \in E=E''$ and $y' \in F'$. If $L(E, F)$ is reflexive, so is $C(E, F)$ so there is T_0 in $C(E, F)$ for which

$$\langle x \otimes y', \phi\rangle = \langle T_0, x \otimes y'\rangle = \langle x, T_0'y'\rangle = \langle T_0 x, y'\rangle \tag{2}$$

for $x \in E$ and $y' \in F'$. Thus $T_0=T$ so that $T \in C(E, F)$.

SUFFICIENCY OF (a) AND (b). We proceed as in the proof of the necessity of (b) to the point of showing that the correspondence from ϕ in $C(E, F)''$ to T_ϕ in $L(E'', F'')$ is one-to-one and onto. Because of (a), $L(E'', F'')=L(E, F)$ and because of (b), $L(E, F)=C(E, F)$. For each ϕ in $C(E, F)''$ there is thus T in $C(E, F)$ with $\langle y'\otimes x, \phi\rangle=\langle Tx, y'\rangle$ for each x in E and y' in F'. But this implies that for $S=\sum_{n=1}^{\infty} x_n\otimes y_n'$ in $I(E', F')$ we have

$$\langle S, \phi\rangle = \sum_{n=1}^{\infty} \langle Tx_n, y_n'\rangle$$

so that $C(E, F)=L(E, F)$ is reflexive.

COROLLARY. *If* $1<q<p<\infty$ *then* $L(l^p, l^q)$ *is reflexive.*

PROOF. By Theorem 1 of [**2**] every bounded linear operator from l^p to l^q is compact.

Other results on compactness of operators from $L^p(\mu)$ to $L^q(\mu)$ can be found in **[3]**, and the interested reader can determine when $L(L^p(\mu), L^q(\mu))$ is reflexive in such situations.

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