REFLEXIVITY OF $L(E, F)$

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Abstract. Let $E$ and $F$ be two Banach spaces both having the approximation property. The space $L(E, F)$ is reflexive if and only if (a) both $E$ and $F$ are reflexive, (b) every continuous linear operator from $E$ into $F$ is compact. Thus $L(l^p, l^q)$ is reflexive for $1 < q < p < \infty$.

In this note we piece together some results of Grothendieck to ascertain when $L(E, F)$, the space of continuous linear operators from a Banach space $E$ to a Banach space $F$, is reflexive. This condition sometimes holds when both $E$ and $F$ have infinite dimension. This demolishes an apparently popular supposition that $L(E, F)$ is reflexive only if one space is reflexive and the other finite dimensional.

The space of compact linear operators from $E$ to $F$ is denoted by $C(E, F)$. It is known that $C(E, F)$ is a closed subspace of $L(E, F)$. A Banach space $E$ is said to have the approximation property if for each Banach space $F$, $C(E, F)$ is the closure in $L(E, F)$ of those operators of finite rank; in other words if $C(E, F)$ is the closed linear span of the set of all one dimensional operators $x' \otimes y$. The value of $x' \otimes y$ with $x' \in E'$, the dual space of $E$, and $y \in F$ is given by the formula

$$x' \otimes y(x) = \langle x, x' \rangle y, \quad x \in E.$$  

Equivalent formulations of the approximation property can be found on pp. 164–165 of [1].

Theorem. Let $E$ and $F$ be two Banach spaces both having the approximation property. The space $L(E, F)$ is reflexive if and only if the following pair of conditions holds:

(a) both $E$ and $F$ are reflexive,
(b) $L(E, F) = C(E, F)$.

Proof. Suppose $\phi$ is in $C(E, F)'$. For $x' \in E'$ the equation

$$\langle T_{\phi} x', y \rangle = \langle x', \otimes y, \phi \rangle, \quad y \in F,$$

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171
determines a linear functional $T_\phi x'$ on $F$. Since $|\langle T_\phi x', y \rangle| \leq \|x' \otimes y\| \|\phi\| = \|x'\| \|y\| \|\phi\|$, $T_\phi x'$ is in $F'$ and $T_\phi \in L(E', F')$ with $\|T_\phi\| \leq \|\phi\|$. If $T_\phi = T_\theta$ then

$$\langle x' \otimes y, \phi \rangle = \langle x' \otimes y, \theta \rangle$$

for each $x'$ in $E'$ and $y$ in $F$. Since $E$ has the approximation property we have $\phi = \theta$. The space of all $T_\phi$ for $\phi$ in $C(E, F)'$ is called the space of integral operators and is denoted by $I(E, F)$. Let $I(E, F)$ be given the topology of identification with $C(E, F)'$.

For $x''$ in $E''$ and $y'$ in $F'$, $x'' \otimes y'$ is in $I(E', F')$ determined by $\phi$ in $C(E, F)$ for which

$$\langle T, \phi \rangle = \langle T'y', x'' \rangle, \quad T \in C(E, F).$$

Here $T'$ denotes the adjoint of $T$. The norm of this $\phi$ is $\|x''\| \|y'\|$, thus $\|x' \otimes y'\|_F = \|x''\| \|y'\|$.

Since $I(E', F')$ is complete it must contain all operators of the form

$$S = \sum_{n=1}^{\infty} x_n'' \otimes y_n'$$

where $\sum_{n=1}^{\infty} \|x_n''\| \|y_n'\| < \infty$. The expansion for $S$ given by (1) is not unique. An operator having such an expansion is called nuclear; the set of all nuclear operators in $I(E', F')$ is denoted by $N(E', F')$. Clearly, $N(E', F')$ is contained in the closed linear span in $I(E', F')$ of all one dimensional operators $x'' \otimes y'$.

Necessity of (a). Let $y_0$ be a vector in $F$ with $\|y_0\| = 1$. The correspondence of $x'$ in $E'$ to $x' \otimes y_0$ in $L(E, F)$ is an isometry for $E'$ onto a subspace of $L(E, F)$. Thus if $L(E, F)$ is reflexive so is $E'$ and hence $E$. A similar proof shows $F$ reflexive.

Necessity of (b). Every $\phi$ in $C(E, F)^*$ can be considered as a continuous linear functional on $I(E', F')$. For each $s''$ in $E''$ we determine $T_\phi x''$ in $F''$ by the rule

$$\langle y', T_\phi x'' \rangle = \langle x'' \otimes y', \phi \rangle, \quad y' \in F'.$$

Since

$$|\langle x'' \otimes y', \phi \rangle| \leq \|x''\| \|y'\| \|\phi\|,$$

it follows that $T_\phi$ is continuous and $\|T_\phi\| \leq \|\phi\|$. The correspondence of $\phi$ in $C(E, F)^*$ to $T_\phi$ in $L(E'', F')$ is linear and continuous. In order to show it is one-to-one we appeal to a result of Grothendieck [1, p. 134] which asserts that if $F'$ is reflexive then $I(E', F') = N(E', F')$ and

$$\|S\|_I = \inf \left\{ \sum_{n=1}^{\infty} \|x_n''\| \|y_n\| : S \text{ has the form (1)} \right\}.$$
If $T_\phi=0$ we then have
\[
\langle y', T_\phi x'' \rangle = \langle x'' \otimes y', \phi \rangle = 0
\]
for each one-dimensional operator $x'' \otimes y'$, from which we conclude $\phi=0$.

The correspondence from $\phi$ in $C(E, F)'$ to $T_\phi$ in $L(E'', F'')$ is also onto when $F'$ is reflexive. For $T$ in $L(E'', F'')$ define $\phi$ on $N(E', F')$ by
\[
\langle \sum_{n=1}^\infty x_n'' \otimes y'_n, \phi \rangle = \sum_{n=1}^\infty \langle Tx_n'', y_n' \rangle.
\]
Since $F''=F$ has the approximation property, the value of $\phi$ does not depend on the representation of $\sum_{n=1}^\infty x_n'' \otimes y'_n$ [1, pp. 164–165]. We then have
\[
\sum_{n=1}^\infty |\langle Tx_n'', y_n' \rangle| \leq \|T\| \sum_{n=1}^\infty \|x_n''\| \|y_n'\|,
\]
so that for each $S$ in $N(E', F')$,
\[
|\langle S, \phi \rangle| \leq \|S\| \|T\|.
\]
Therefore $\phi$ is continuous on $I(E', F')$ and $T=T_\phi$.

For each $T$ in $L(E'', F'')=L(E, F)$ there is $\phi$ in $C(E, F)'$ such that $\langle x' \otimes y', \phi \rangle = \langle Tx, y' \rangle$ for each $x \in E=E''$ and $y' \in F'$. If $L(E, F)$ is reflexive, so is $C(E, F)$ so there is $T_0$ in $C(E, F)$ for which
\[
\langle x \otimes y', \phi \rangle = \langle T_0, x \otimes y' \rangle = \langle x, T_0 y' \rangle = \langle T_0 x, y' \rangle
\]
for $x \in E$ and $y' \in F'$. Thus $T_0=T$ so that $T \in C(E, F)$.

**Sufficiency of (a) and (b).** We proceed as in the proof of the necessity of (b) to the point of showing that the correspondence from $\phi$ in $C(E, F)'$ to $T_\phi$ in $L(E'', F'')$ is one-to-one and onto. Because of (a), $L(E'', F'')=L(E, F)$ and because of (b), $L(E, F)=C(E, F)$. For each $\phi$ in $C(E, F)'$ there is thus $T$ in $C(E, F)$ with $\langle y' \otimes x, \phi \rangle = \langle Tx, y' \rangle$ for each $x \in E$ and $y' \in F'$. But this implies that for $S=\sum_{n=1}^\infty x_n \otimes y_n'$ in $I(E', F')$ we have
\[
\langle S, \phi \rangle = \sum_{n=1}^\infty \langle Tx_n, y_n' \rangle
\]
so that $C(E, F)=L(E, F)$ is reflexive.

**Corollary.** If $1<q<p<\infty$ then $L(p, l^q)$ is reflexive.

**Proof.** By Theorem 1 of [2] every bounded linear operator from $l^p$ to $l^q$ is compact.
Other results on compactness of operators from $L^p(\mu)$ to $L^q(\mu)$ can be found in [3], and the interested reader can determine when $L(L^p(\mu), L^q(\mu))$ is reflexive in such situations.

REFERENCES


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