REFLEXIVITY OF $L(E, F)$

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Abstract. Let $E$ and $F$ be two Banach spaces both having the approximation property. The space $L(E, F)$ is reflexive if and only if (a) both $E$ and $F$ are reflexive, (b) every continuous linear operator from $E$ into $F$ is compact. Thus $L(l^p, l^q)$ is reflexive for $1 < q < p < \infty$.

In this note we piece together some results of Grothendieck to ascertain when $L(E, F)$, the space of continuous linear operators from a Banach space $E$ to a Banach space $F$, is reflexive. This condition sometimes holds when both $E$ and $F$ have infinite dimension. This demolishes an apparently popular supposition that $L(E, F)$ is reflexive only if one space is reflexive and the other finite dimensional.

The space of compact linear operators from $E$ to $F$ is denoted by $C(E, F)$. It is known that $C(E, F)$ is a closed subspace of $L(E, F)$. A Banach space $E$ is said to have the approximation property if for each Banach space $F$, $C(E, F)$ is the closure in $L(E, F)$ of those operators of finite rank; in other words if $C(E, F)$ is the closed linear span of the set of all one dimensional operators $x' \otimes y$. The value of $x' \otimes y$ with $x' \in E'$, the dual space of $E$, and $y \in F$ is given by the formula

$$x' \otimes y(x) = \langle x', x' \rangle y, \quad x \in E.$$ 

Equivalent formulations of the approximation property can be found on pp. 164-165 of [1].

Theorem. Let $E$ and $F$ be two Banach spaces both having the approximation property. The space $L(E, F)$ is reflexive if and only if the following pair of conditions holds:

(a) both $E$ and $F$ are reflexive,
(b) $L(E, F) = C(E, F)$.

Proof. Suppose $\phi$ is in $C(E, F)'$. For $x' \in E'$ the equation

$$\langle T_\phi x', y \rangle = \langle x' \otimes y, \phi \rangle, \quad y \in F,$$

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determines a linear functional $T_\phi x'$ on $F$. Since $|\langle T_\phi x', y \rangle| \leq \|x' \otimes y\| \|\phi\| = \|x'\| \|y\| \|\phi\|$, $T_\phi x'$ is in $F'$ and $T_\phi \in L(E', F')$ with $\|T_\phi\| \leq \|\phi\|$. If $T_\phi = T_\theta$ then
\[ \langle x' \otimes y, \phi \rangle = \langle x' \otimes y, \theta \rangle \]
for each $x'$ in $E'$ and $y$ in $F$. Since $E$ has the approximation property we have $\phi = 0$. The space of all $T_\phi$ for $\phi$ in $C(E, F)'$ is called the space of integral operators and is denoted by $I(E, F)$. Let $I(E, F)$ be given the topology of identification with $C(E, F)'$.

For $x''$ in $E''$ and $y'$ in $F'$, $x'' \otimes y'$ is in $I(E', F')$ determined by $\phi$ in $C(E, F)$ for which
\[ \langle T, \phi \rangle = \langle T'y', x'' \rangle, \quad T \in C(E, F). \]
Here $T'$ denotes the adjoint of $T$. The norm of this $\phi$ is $\|x''\| \|y'\|$; thus $\|x' \otimes y'\|_I = \|x''\| \|y'\|$.

Since $I(E', F')$ is complete it must contain all operators of the form
\[ S = \sum_{n=1}^{\infty} x_n' \otimes y_n' \]
where $\sum_{n=1}^{\infty} \|x_n'\| \|y_n'\| < \infty$. The expansion for $S$ given by (1) is not unique. An operator having such an expansion is called nuclear; the set of all nuclear operators in $I(E', F')$ is denoted by $N(E', F')$. Clearly, $N(E', F')$ is contained in the closed linear span in $I(E', F')$ of all one dimensional operators $x' \otimes y'$.

**Necessity of (a).** Let $y_0$ be a vector in $F$ with $\|y_0\| = 1$. The correspondence of $x'$ in $E'$ to $x' \otimes y_0$ in $L(E, F)$ is an isometry for $E'$ onto a subspace of $L(E, F)$. Thus if $L(E, F)$ is reflexive so is $E'$ and hence $E$. A similar proof shows $F$ reflexive.

**Necessity of (b).** Every $\phi$ in $C(E, F)''$ can be considered as a continuous linear functional on $I(E', F')$. For each $s''$ in $E''$ we determine $T_\phi x''$ in $F''$ by the rule
\[ \langle y', T_\phi x'' \rangle = \langle x'' \otimes y', \phi \rangle, \quad y' \in F'. \]
Since
\[ |\langle x'' \otimes y', \phi \rangle| \leq \|x''\| \|y'\| \|\phi\|, \]
it follows that $T_\phi$ is continuous and $\|T_\phi\| \leq \|\phi\|$. The correspondence of $\phi$ in $C(E, F)''$ to $T_\phi$ in $L(E'', F')$ is linear and continuous. In order to show it is one-to-one we appeal to a result of Grothendieck [1, p. 134] which asserts that if $F'$ is reflexive then $I(E', F') = N(E', F')$ and
\[ \|S\|_I = \inf \left\{ \sum_{n=1}^{\infty} \|x_n''\| \|y_n\| : S \text{ has the form (1)} \right\}. \]
If \( T_\phi = 0 \) we then have
\[
\langle y', T_\phi x'' \rangle = \langle x'' \otimes y', \phi \rangle = 0
\]
for each one-dimensional operator \( x'' \otimes y' \), from which we conclude \( \phi = 0 \).

The correspondence from \( \phi \) in \( C(E, F)'' \) to \( T_\phi \) in \( L(E'', F'') \) is also onto when \( F' \) is reflexive. For \( T \) in \( L(E'', F'') \) define \( \phi \) on \( N(E', F') \) by
\[
\langle \sum_{n=1}^{\infty} x_n'' \otimes y'_n, \phi \rangle = \sum_{n=1}^{\infty} \langle Tx_n'', y'_n \rangle.
\]
Since \( F'' = F \) has the approximation property, the value of \( \phi \) does not depend on the representation of \( \sum_{n=1}^{\infty} x''_n \otimes y'_n \) [1, pp. 164–165]. We then have
\[
\sum_{n=1}^{\infty} |\langle Tx_n'', y'_n \rangle| \leq \|T\| \sum_{n=1}^{\infty} \|x''_n\| \|y'_n\|,
\]
so that for each \( S \) in \( N(E', F') \),
\[
|\langle S, \phi \rangle| \leq \|S\| \|T\|.
\]
Therefore \( \phi \) is continuous on \( I(E', F') \) and \( T = T_\phi \).

For each \( T \) in \( L(E'', F'') = L(E, F) \) there is \( \phi \) in \( C(E, F)'' \) such that \( \langle x' \otimes y', \phi \rangle = \langle Tx, y' \rangle \) for each \( x \in E = E'' \) and \( y' \in F' \). If \( L(E, F) \) is reflexive, so is \( C(E, F) \) so there is \( T_0 \) in \( C(E, F) \) for which
\[
\langle x \otimes y', \phi \rangle = \langle T_0, x \otimes y' \rangle = \langle x, T_0' y' \rangle = \langle T_0 x, y' \rangle
\]
for \( x \in E \) and \( y' \in F' \). Thus \( T_0 = T \) so that \( T \in C(E, F) \).

**Sufficiency of (a) and (b).** We proceed as in the proof of the necessity of (b) to the point of showing that the correspondence from \( \phi \) in \( C(E, F)'' \) to \( T_\phi \) in \( L(E'', F'') \) is one-to-one and onto. Because of (a), \( L(E'', F'') = L(E, F) \) and because of (b), \( L(E, F) = C(E, F) \). For each \( \phi \) in \( C(E, F)'' \) there is thus \( T \) in \( C(E, F) \) with \( \langle y' \otimes x, \phi \rangle = \langle Tx, y' \rangle \) for each \( x \) in \( E \) and \( y' \) in \( F' \). But this implies that for \( S = \sum_{n=1}^{\infty} x_n \otimes y'_n \) in \( I(E', F') \) we have
\[
\langle S, \phi \rangle = \sum_{n=1}^{\infty} \langle Tx_n, y'_n \rangle
\]
so that \( C(E, F) = L(E, F) \) is reflexive.

**Corollary.** If \( 1 < q < p < \infty \) then \( L(p, l^q) \) is reflexive.

**Proof.** By Theorem 1 of [2] every bounded linear operator from \( l^p \) to \( l^q \) is compact.
Other results on compactness of operators from $L^p(\mu)$ to $L^q(\mu)$ can be found in [3], and the interested reader can determine when $L(L^p(\mu), L^q(\mu))$ is reflexive in such situations.

REFERENCES


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