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NONASYMPTOTICALLY ABELIAN FACTORS OF TYPE III

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ABSTRACT. There exists a continuum of nonisomorphic nonasymptotically abelian type III factors on separable Hilbert spaces.

1. A von Neumann algebra R is called *asymptotically abelian* if there exists a sequence $\{\alpha_n\}$ of *-automorphisms of R such that for any S, $T \in R$, $[\alpha_n(S), T] \to 0$ strongly, where $[A, B] = AB - BA$, i.e. $\{\alpha_n(S)\}$ is a central sequence[2] for each $S \in R$. This definition was introduced by Sakai in **[6]**, where it was proved that no finite type I factor is asymptotically abelian, and that there exist asymptotically abelian and nonasymptotically abelian factors of type II_1, both having nontrivial central sequences. In [7] Willig proved that the type I_∞ factor is not asymptotically abelian. Glaser **[4]** showed that no type II_∞ factor is asymptotically abelian, and also applied the method of [7] to show that one example of type III factors in Pukanszky **[5]** is not asymptotically abelian. Indeed, this particular factor has no nontrivial central sequence. In this note we shall show that the continuum of nonisomorphic type III factors constructed in **[2]** consists entirely of nonasymptotically abelian factors each with nontrivial central sequences.

2. Let R be a von Neumann algebra. A von Neumann subalgebra B of R is called a *residual subalgebra* of R if for any central sequence $\{T_n\}$ in R there exists a bounded sequence $\{S_n\}$ in B such that $T_n - S_n \to 0$ strongly.

THEOREM. *Let R be a properly infinite von Neumann algebra on a Hilbert space H. Let $x \in H$ be a unit vector, and B a residual subalgebra of R be such that $(STx|x) = (TSx|x)$ for all $T \in R$ and $S \in B$. Then R is not asymptotically abelian.*

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[2] A bounded sequence $\{T_n\}$ of operators in a von Neumann algebra R is called *central* if $[T_n, T] \to 0$ strongly for all $T \in R$. The central sequence $\{T_n\}$ is called *trivial* if there exists a sequence $\{c_n\}$ of complex numbers such that $T_n - c_n I \to 0$ strongly. See Dixmier and Lance [3].

PROOF. Suppose that R is asymptotically abelian, and that $\{\alpha_n\}$ is the requisite sequence of *-automorphisms of R. Define a sequence of functionals on R by

$$\phi_n(T) = (\alpha_n(T)x \mid x), \qquad T \in R.$$

Since $\|\phi_n\|=1$, by the weak*-compactness of the unit ball of the dual of R, there exists a subsequence of $\{\phi_n\}$ (which we again call $\{\phi_n\}$) converging weakly to some ϕ_0 in the unit ball. Now for any T, $S \in R$, $\{\alpha_n(T)\}$ and $\{\alpha_n(S)\}$ are central sequences, and hence there exist sequences $\{T_n\}$ and $\{S_n\}$ in B such that $\alpha_n(T)-T_n \to 0$ strongly and $\alpha_n(S)-S_n \to 0$ strongly. Since strong convergence of operators is jointly continuous on bounded sets, it follows that

$$\alpha_n(T)\alpha_n(S) - \alpha_n(T)S_n - T_n\alpha_n(S) + T_nS_n \to 0 \quad \text{strongly.}$$

Now note that

$$\begin{aligned}\phi_0(TS) &= \lim_n(\alpha_n(T)\alpha_n(S)x \mid x) = \lim_n((\alpha_n(T)S_n + T_n\alpha_n(S) - T_nS_n)x \mid x) \\ &= \lim_n((S_n\alpha_n(T) + \alpha_n(S)T_n - S_nT_n)x \mid x) = \phi_0(ST).\end{aligned}$$

Thus ϕ_0 is a finite trace on R, contradicting the assumption that R is properly infinite. Q.E.D.

COROLLARY. *There exists a continuum of nonisomorphic nonasymptotically abelian factors of type* III.

PROOF. Let $R_1=M(X, \mu)\otimes\Delta$ be the factor of type III on the Hilbert space $H_1=L^2(X, \mu)\otimes l^2(\Delta)$ with $1\otimes\delta_e$ as a separating cyclic vector, as constructed in §3 of [**1**] (or see [**5**]). Let $\{G_\alpha\}_{\alpha\in I}$ be the continuum of groups constructed in §2 of [**2**]. Let $R_\alpha=R_1 \otimes_\phi G_\alpha$ be the factor of type III on the Hilbert space $H=H_1\otimes l^2(G_\alpha)$ with a separating cyclic unit vector $\xi= 1\otimes\delta_e\otimes\delta_e$. By Remark 4 of [**2**], $B_\alpha=M(X, \mu)\otimes\mathscr{A}(H_N)$ is residual in R_α. As observed in the proof of that remark, we have

$$(TS\xi \mid \xi) = (ST\xi \mid \xi) \quad \text{for any } T \in R_\alpha,\ S \in B_\alpha.$$

Hence by the theorem above and by Theorem 1 in [**2**], $\{R_\alpha\}_{\alpha\in\tilde{I}}$ is a continuum of pairwise nonisomorphic nonasymptotically abelian factors of type III. Q.E.D.

REMARK 1. If R is properly infinite and has only trivial central sequences, then take $B=\boldsymbol{C}$, the field of complex numbers, and x any unit vector in the above Theorem, so that R is not asymptotically abelian. Thus any asymptotically abelian properly infinite von Neumann algebra must have nontrivial central sequence. It is worth noting, in this regard, that the factors $\{R_\alpha\}_{\alpha\in\tilde{I}}$ all have nontrivial central sequences.

REMARK 2. By the same proof as in the Corollary, the factors R_3 and R_4 constructed in [**1**] are not asymptotically abelian.

Finally, we remark that it is not known whether the corresponding II_1-factors $\{\mathscr{A}(G_\alpha)\}_{\alpha\in I}$ constructed in [**2**] are asymptotically abelian or not.

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