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ORTHOGONALITY AND NONLINEAR FUNCTIONALS ON BANACH SPACES

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ABSTRACT. If B is a real Banach space and $x, y \in B$, then x is said to be orthogonal to y ($x \perp y$) if $\|x+\lambda y\| \geqq \|x\|$ for all real numbers λ. A function $F: B \to E$, where E is a topological vector space, is said to be additive if it is continuous and $F(x+y)=F(x)+F(y)$ whenever $x \perp y$. The purpose of the present paper is to characterize additive functions.

If M is a Banach space of real valued measurable functions on a measure space and if $x, y \in M$, then x is said to be orthogonal to y in the lattice theoretic sense ($x \perp_L y$), if the set $\{t | x(t)y(t) \neq 0\}$ is of measure zero. A real valued function F on M is said to be L-additive if it is continuous, and $F(x+y)=F(x)+F(y)$ whenever $x \perp_L y$. Integral representations of L-additive functionals have been the subject of extensive study in recent years. For these and related results we refer to Drewnowski and Orlicz [**1**] and Sundaresan [**2**], and the bibliography cited therein. The concept of orthogonality in the definition of L-additive functionals on M is very natural in these spaces. However there are several other concepts of orthogonality which have been studied in detail in arbitrary Banach spaces; see, for example, James ([**3**], [**4**]). These concepts are generalizations of orthogonality in Euclidean spaces and are of intrinsic geometric interest. One such concept of orthogonality is as follows. If B is a real Banach space, and $x, y \in B$, then x is said to be orthogonal to y, in short $x \perp y$, if $\|x+\lambda y\| \geqq \|x\|$ for all real numbers λ. The purpose of the present note is to characterize continuous real valued functions F on B such that $F(x+y)=F(x)+F(y)$, whenever $x \perp y$. For a motivation of the study of such functionals it is enough to note that for $x, y \in L_p(\mu)$, $x \perp_L y$ implies $x \perp y$, while in general the implication cannot be reversed.

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In the rest of the paper B stands for an arbitrary real Banach space of dimension at least 2. Before proceeding to the main results of the paper we summarize useful facts concerning the concept of orthogonality ($\perp$) described above. (1) Orthogonality is homogeneous, i.e., $x \perp y \Rightarrow \lambda x \perp \mu y$ for all real numbers λ and μ. (2) Orthogonality is not symmetric in general, i.e., $x \perp y$ does not necessarily imply $y \perp x$. However it is known, Day **[5]**, that if the dimension of $B \geqq 3$, then orthogonality is symmetric if and only if B is isometric with a Hilbert space. (3) If $x \in B$ then there exists a nonzero vector $y \in B$ such that $x \perp y$.

In passing we note that if F is a continuous additive functional on B, then the symmetric and antisymmetric parts F_1, F_2 of F are also continuous additive functionals on B. This is verified from the equations $F_1(x) = \frac{1}{2}[F(x)+F(-x)]$, and $F_2(x)=\frac{1}{2}[F(x)-F(-x)]$.

We start noting three useful lemmas.

LEMMA 1. *If B is two-dimensional, then there are nonzero vectors u and v such that $u \perp v$ and $u+v \perp u-v$.*

PROOF. Choose arbitrary vectors x and y such that $x \perp y$, and $\|x\| = \|y\|=1$. Represent B in the plane so that $x \leftrightarrow (1, 0)$, $y \leftrightarrow (0, 1)$, and $ax+by \leftrightarrow (a, b)$. Since $x \perp y$, there is a vertical line of support to the unit circle at $(1, 0)$. As this line of support rotates counterclockwise, its point of support is at $(x+ay)/\|x+ay\|$, where a is a continuous function of the angle of inclination. The ray from 0 to $x-ay$ rotates from horizontal to the vertical, clockwise, as a varies from 0 to ∞. Therefore there is a value of a and a line of support at the corresponding point $(x+ay)/\|x+ay\|$ that is parallel to the ray from 0 to $x-ay$. That is, $x+ay \perp x-ay$. Now let $u=x$ and $v=ay$.

LEMMA 2. *Let B be a Banach space. If F is an antisymmetric continuous additive functional on B, then F is linear.*

PROOF. Let x and y be any two members of B. Choose nonzero vectors u and v in the linear span of x and y such that $u \perp v$ and $u+v \perp u-v$. By considering $F[\lambda(u+v) \pm \lambda(u-v)]$, we obtain

$$F(2\lambda u) = F[\lambda(u+v)] + F[\lambda(u - v)] = 2F(\lambda u),$$

and $F(2\lambda v)=2F(\lambda v)$. Then by letting $\lambda=\frac{1}{2}\nu$, we obtain that $F(\frac{1}{2}\nu u)= \frac{1}{2}F(\nu u)$. By induction, it follows that $F(2^{p/q}u)=2^{p/q}F(u)$ for all positive integers p and q. It now follows from the continuity and antisymmetry of F that $F(au)=aF(u)$ for all real numbers a. Similarly $F(bv)=bF(v)$. Thus $F(au+bv)=F(au)+F(bv)=aF(u)+bF(v)$ so that F is linear on the span of u and v. Thus $F(ax+by)=aF(x)+bF(y)$, for all real numbers a and b. Hence F is a linear functional on B.

LEMMA 3. *Let B be a Banach space and let F be a symmetric additive functional on B. If B is not isometric to a Hilbert space, then $F=0$. If B is isometric to a Hilbert space, then there is a real number c such that, for all $x \in B$, $F(x)=c\|x\|^2$.*

PROOF. Let u and v be any two members of B such that $u\neq 0$, $u \perp v$, and $u+v \perp u-v$. Then we obtain from $F[\lambda(u+v)\pm\lambda(u-v)]$ the two equations:

$$F(2\lambda u) = 2F(\lambda u) + 2F(\lambda v), \qquad F(2\lambda v) = 2F(\lambda u) + 2F(\lambda v).$$

These equations imply that $F(2\lambda u)=F(2\lambda v)$ for all real numbers λ. This and the first equation imply that $F(2\lambda u)=4F(\lambda u)$. Then also $F(\frac{1}{2}\nu u)=\frac{1}{4}F(\nu u)$ for all ν, and induction as in Lemma 2 gives

$$F(au) = a^2F(u), \quad \text{for all } a.$$

Similarly, $F(bv)=b^2F(v)$, for all b. Thus since $u \perp v$, $F(au+bv)=a^2F(u)+b^2F(v)$. Since $\lambda=\frac{1}{2}$ gives $F(u)=F(v)$, it is verified that

$$F(au + bv) = a^2F(u) + b^2F(v). \tag{1}$$

Suppose first that B is isometric to a Hilbert space. Let $\{e_\alpha\}$ be an orthonormal basis of B. Since $e_\alpha+e_\beta \perp e_\alpha-e_\beta$ for all α and β, it follows from (1) that if $e=F(e_\alpha)$ for some specific α and if $x=\sum a_ie_{\beta(i)}$, then

$$F(x) = F(\textstyle\sum a_ie_{\beta(i)}) = \sum (a_i)^2F(e_\alpha) = \|x\|^2e.$$

Next suppose that B is not isometric with a Hilbert space. Then there is a two-dimensional subspace E that is not isometric to Hilbert space. Now Lemma 1 assures the existence of two vectors u, v such that $u \perp v$, $u+v \perp u-v$, and the linear span of u, v is E. Let now E be represented in the plane by $u\leftrightarrow(1, 0)$ and $v\leftrightarrow(0, 1)$. Then there is a line of support L to the unit circle at a point $(\alpha u+\beta v)/\|\alpha u+\beta v\|$, for which L is not perpendicular to the ray from the origin to $\alpha u+\beta v$. Thus $au+\beta v \perp \beta u-cv$ where $c\neq\alpha$, and the rays from the origin to (α, β) and $(\beta, -c)$ are not perpendicular. Then

$$\begin{aligned} F[(c\alpha + \beta^2)u] &= F[c(\alpha u + \beta v) + \beta(\beta u - cv)] \\ &= F(c\alpha u + c\beta v) + F(\beta^2 u - \beta cv). \end{aligned}$$

It then follows from (1) that $(c\alpha+\beta^2)^2=c^2\alpha^2+c^2\beta^2+\beta^4+\beta^2c^2$ and $c\alpha\beta^2=c^2\beta^2$. Similarly, $c\alpha\beta^2=\alpha^2\beta^2$. Since $c\neq\alpha$, it follows that $\beta=0$. This is impossible, since $(\alpha, 0)$ is not orthogonal to $(0, -c)$ unless $c=0$. This completes the proof of the lemma.

We now proceed to the main result of the paper characterizing continuous additive functionals on a Banach space. We state the result in the more general setting, allowing the additive function to take values in a locally convex space.

THEOREM 1. *Let F be a continuous function on a Banach space B into a locally convex space E. If B is a Hilbert space, then F is additive if and only if there exist a vector $\xi \in E$ and a continuous linear operator $T: B \to E$ such that $F(x) = \|x\|^2 \xi + T(x)$ for all $x \in E$. Further if B is not isometric with a Hilbert space, then F is a continuous linear operator on B to E.*

PROOF. Let B be a Hilbert space. Let E^* be the topological dual of E. Let F be an additive function on $B \to E$. Let F_1, F_2 be the symmetric and antisymmetric parts of F. Now if $f \in E^*$, then $f \circ F_1$ and $f \circ F_2$ are symmetric and antisymmetric additive functionals on B. Thus from Lemmas 2 and 3, it is inferred that there are a constant c_f and a continuous linear functional l_f on B such that $f \circ F_1(x) = c_f \|x\|^2$ and $f \circ F_2(x) = l_f(x)$. It is verified that the mapping $f \to c_f$ is an ω^*-continuous linear functional on E^*. Hence there exists a fixed vector $\xi \in E$ such that $f(\xi) = c_f$ for all $f \in E^*$. Since E^* separates points in E, it follows that $F_1(x) = \|x\|^2 \xi$. Again, since for each $f \in E^*$, $f \circ F_2$ is a linear functional on B, it follows that F_2 is a linear operator on $B \to E$. Further, since F is continuous, F_2 is a continuous linear operator on $B \to E$. Considering symmetric and antisymmetric parts of F it is verified that ξ and T are uniquely determined by F. Thus $F(x) = \|x\|^2 \xi + T(x)$ as stated in the theorem. Since any function F with the above representation is an additive function, the proof of the part of the theorem dealing with the case when B is a Hilbert space is complete. The case when B is not a Hilbert space is similarly dealt with, and the proof is omitted.

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