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# CRITERIA FOR DISCRETE SPECTRUM OF SINGULAR SELFADJOINT DIFFERENTIAL OPERATORS

L. W. ROLLINS

ABSTRACT. Under certain conditions on the coefficients of symmetric singular differential operators of order $2n$, selfadjoint extensions are shown to have a discrete spectrum. The results are proven specifically for the Friedrichs extension.

1. **Introduction.** Consider the formally selfadjoint differential operator defined by

$$\tau u(x) = \frac{1}{m(x)} \sum_{k=0}^{n} (-1)^k D^k p_k(x) D^k u(x), \qquad a \leqq x < b,$$

where $b$ may be finite or infinite, $p_k$ is real-valued, nonnegative, and in $C^{(k)}[a, b)$ for $0 \leqq k \leqq n$, and, for $x \in [a, b)$, $p_n(x) > 0$, and $m(x) > 0$. The endpoint $b$ is possibly a singular endpoint.

Let $L^2(m)$ be the Hilbert space of all measurable functions, $f$, defined on $(a, b)$ for which $\int_a^b |f(x)|^2 m(x)\, dx < \infty$, with inner product and norm denoted by $(\cdot, \cdot)$ and $\|\cdot\|$ respectively. Let $T_0$ be the symmetric linear operator defined by $T_0 u = \tau u$ whose domain $D(T_0)$ consists of those functions in $L^2(m)$ which have $2n$ continuous derivatives and compact support in $(a, b)$. Then $D(T_0)$ is dense in $L^2(m)$. Under these conditions $T_0$ will have a selfadjoint extension; in particular, it will have the Friedrichs extension [**6**, p. 124].

This paper is concerned with finding conditions on the functions $p_k$, $1 \leqq k \leqq n$, which guarantee that the Friedrichs extension has a compact inverse and, hence, a discrete spectrum. Since all selfadjoint extensions of $T_0$ have the same continuous spectrum [**5**, Theorem 2, p. 92], we conclude that every selfadjoint extension of $T_0$ has a discrete spectrum.

Early investigations were confined to second order differential operators with the weight function $m(x) = 1$ and are available in the literature (see for example: Naĭmark [**5**, Chapter VII], and Dunford-Schwartz [**2**, Chapter XIII]). Recently Eastham [**3**] obtained results for higher order

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differential operators with $b=\infty$. The results here will be compared with Eastham's and, also, earlier results of Friedrichs [4]. The techniques are an extension of those used by Baxley [1].

2. **Statement of theorem.** The operator $T_0$ is symmetric and semi-bounded from below. To see the latter, use repeated integration by parts and the fact that $p_k \geqq 0$ for $0 \leqq k \leqq n$, to get

$$(T_0u, u) = \int_a^b \sum_{k=0}^n p_k(x)[u^{(k)}(x)]^2\, dx \geqq 0$$

for $u \in D(T_0)$. Therefore $T_0$ has a Friedrichs extension, $T$.

Let $k$ be an integer, $1 \leqq k \leqq n$. For this $k$ define a sequence of functions $\{h_i\}_1^k$ as follows:

If, for each $y$ in $[a, b)$, it is true that $\int_y^b 1/p_k(t)\, dt$ is finite, then

$$h_1(x, k) = \left(\int_x^b \frac{1}{p_k(t)}\, dt\right)^{1/2}, \quad \text{for } x \in [a, b). \tag{1a}$$

Otherwise

$$h_1(x, k) = \left(\int_a^x \frac{1}{p_k(t)}\, dt\right)^{1/2}, \quad \text{for } x \in [a, b). \tag{1b}$$

If $i$ is any integer $1 \leqq i \leqq k-1$, and if, for each $y$ in $[a, b)$, it is true that $\int_y^b h_i(t, k)\, dt$ is finite, then

$$h_{i+1}(x, k) = \int_x^b h_i(t, k)\, dt, \quad \text{for } x \in [a, b). \tag{2a}$$

Otherwise

$$h_{i+1}(x, k) = \int_a^x h_i(t, k)\, dt, \quad \text{for } x \in [a, b). \tag{2b}$$

THEOREM. *Let $T$ be the Friedrichs extension of $T_0$. If*

$$\int_a^b m(x)[h_k(x, k)]^2\, dx = M < \infty \tag{3}$$

*for at least one integer $k$, $1 \leqq k \leqq n$, then $T$ has a compact inverse, and in this case, every selfadjoint extension of $T_0$ has a discrete spectrum.*

3. **Proof of theorem.** We first prove two lemmas.

LEMMA 1. *If $u \in D(T_0)$ and* (3) *is satisfied for some $k$, $1 < k \leqq n$, then for $a \leqq x_1 < x_2 < b$,*

$$|u(x_2) - u(x_1)| \leqq (T_0u, u)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t, k)\, dt.$$

PROOF. For $u \in D(T_0)$, it follows from the Schwartz inequality

$$
\begin{aligned}
|u^{(k-1)}(x_2) - u^{(k-1)}(x_1)|^2 &\leqq \left| \int_{x_1}^{x_2} u^{(k)}(t)\,dt \right|^2 \\
&\leqq \int_{x_1}^{x_2} p_k(t)[u^{(k)}(t)]^2\,dt \int_{x_1}^{x_2} \frac{1}{p_k(t)}\,dt \\
&\leqq (T_0 u, u) \int_{x_1}^{x_2} \frac{1}{p_k(t)}\,dt.
\end{aligned}
$$

Now setting $x_1 = x$ and $x_2 = b$, or $x_1 = a$ and $x_2 = x < b$ for $h_1(x, k)$ defined as in (1a) or (1b) respectively, leads to the following:

$$
\begin{aligned}
|u^{(k-1)}(x)|^2 &\leqq (T_0 u, u)[h_1(x, k)]^2, \\
|u^{(k-1)}(x)| &\leqq (T_0 u, u)^{1/2} h_1(x, k).
\end{aligned} \tag{4}
$$

Using (4) above, we obtain

$$
\begin{aligned}
|u^{(k-2)}(x_2) &- u^{(k-2)}(x_1)| \\
&= \left| \int_{x_1}^{x_2} u^{(k-1)}(t)\,dt \right| \\
&\leqq \left( \int_{x_1}^{x_2} (T_0 u, u)^{1/2} h_1(t, k)\,dt \right) = (T_0 u, u)^{1/2} \left( \int_{x_1}^{x_2} h_1(t, k)\,dt \right).
\end{aligned}
$$

Setting $x_1 = x$ and $x_2 = b$, or $x_1 = a$ and $x_2 = x < b$ for $h_2(x, k)$ defined as in (2a) or (2b) respectively,

$$
\begin{aligned}
|u^{(k-2)}(x)|^2 &\leqq (T_0 u, u)[h_2(x, k)]^2, \\
|u^{(k-2)}(x)| &\leqq (T_0 u, u)^{1/2} h_2(x, k).
\end{aligned}
$$

Continuing by induction, we get the inequality

$$
|u^{(k-j)}(x)| \leqq (T_0 u, u)^{1/2} h_j(x, k), \qquad j = 1, \cdots, k-1,
$$

in particular, for $j = k-1$,

$$
|u'(x)| \leqq (T_0 u, u)^{1/2} h_{k-1}(x, k).
$$

Integration of both sides leads to the conclusion of the lemma.

LEMMA 2. *Let $T$ be the Friedrichs extension of $T_0$. If $u \in D(T)$ and* (3) *holds for some $k$, $1 < k \leqq n$, then for $a \leqq x_1 < x_2 < b$,*

$$
|u(x_2) - u(x_1)| \leqq (Tu, u)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t, k)\,dt, \tag{a}
$$

$$
(u, u) \leqq (Tu, u)M. \tag{b}
$$

Proof. For $u \in D(T)$ there exists a sequence $\{u_m\}$ with $u_m \in D(T_0)$, $m=1, \cdots$, such that $\|u_m - u\| \to 0$ and $(T_0 u_m, u_m) \to (Tu, u)$ as $m \to \infty$.

Since $(T_0 u_m, u_m)$ is a convergent sequence it is bounded, that is $(T_0 u_m, u_m) \leqq C$ for some positive number $C$, $m=1, \cdots$. From Lemma 1,

$$|u_m(x_2) - u_m(x_1)| \leqq (T_0 u_m, u_m)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t, k)\, dt. \tag{5}$$

Therefore the functions $\{u_m\}$ are equicontinuous and uniformly bounded on compact subsets of $[a, b)$. By the Ascoli Theorem there is a subsequence of $\{u_m\}$ which converges uniformly on compact subsets of $[a, b)$. Restricting attention to this subsequence and taking limits in (5) we get (a),

$$|u(x_2) - u(x_1)| \leqq (Tu, u)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t, k)\, dt.$$

For the case $h_k(x, k) = \int_a^x h_{k-1}(t, k)\, dt$, let $x_1 = a$, $x_2 = x$ in the above inequality to obtain

$$|u(x)| \leqq (Tu, u)^{1/2} h_k(x, k).$$

For the case $h_k(x, k) = \int_x^b h_{k-1}(t, k)\, dt$, then note that for each $u_m$ in the subsequence there is a $x_2 < b$ such that $u_m(x_2) = 0$. Setting $x_1 = x$ we obtain

$$\begin{aligned} |u_m(x)| &= |u_m(x_2) - u_m(x)| \\ &\leqq (T_0 u_m, u_m)^{1/2} \int_x^{x_2} h_{k-1}(t, k)\, dt \leqq (T_0 u_m, u_m)^{1/2} \int_x^b h_{k-1}(t, k)\, dt \\ &= (T_0 u_m, u_m)^{1/2} h_k(x, k). \end{aligned}$$

Now take limits to obtain, for this case also, $|u(x)| \leqq (Tu, u)^{1/2} h_k(x, k)$. Hence, in either case

$$|u(x)|^2 \leqq (Tu, u)[h_k(x, k)]^2. \tag{6}$$

Integration of the above leads to (b).

Proof of the Theorem. Let $\lambda$ be an eigenvalue of $T$ associated with eigenvector $u$. Then $(Tu, u) = (\lambda u, u) \geqq 1/M(u, u)$. Hence, $\lambda \geqq 1/M > 0$. Therefore all eigenvalues of $T$ are positive and $T^{-1}$ exists.

Let $\{Tu_n\}$ be a bounded sequence in the domain of $T^{-1}$ and $K$ be a number such that $\|Tu_n\| \leqq K < \infty$, $n=1, \cdots$. Then $u_n \in D(T)$ and, from Lemma 2(b),

$$\|u_n\|^2 = (u_n, u_n) \leqq M(Tu_n, u_n) \leqq M\, \|Tu_n\|\, \|u_n\|,$$

from which follows $\|u_n\| \leqq MK$, and $(Tu_n, u_n) \leqq MK^2$.

From Lemma 2(a),

$$|u_n(x_2) - u_n(x_1)| \leqq (MK^2)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t, k)\, dt$$

and, hence, $\{u_n\}$ is uniformly bounded and equicontinuous on compact subsets of $[a, b)$. Using the Ascoli Theorem we get a subsequence $\{v_n\}$ which converges uniformly on compact subsets of $[a, b)$ to a limit function $u$.

It remains to show that $\{v_n\}$ converges in $L^2(m)$. From (6) above

$$\begin{aligned} m(x)\,|v_n(x)|^2 &\leqq (Tv_n, v_n)m(x)[h_k(x, k)]^2 \\ &\leqq MK^2 m(x)[h_k(x, k)]^2, \qquad n = 1, \cdots. \end{aligned}$$

The right side is integrable by hypothesis. The Lebesgue Dominated Convergence Theorem yields $u \in L^2(m)$ and $\|v_n\| \to \|u\|$. Hence,

$$\|v_n - u\| \to 0.$$

Hence, $T^{-1}$ is compact, and $T$ has a discrete spectrum. It follows [**5**, Theorem 2, p. 92] that every selfadjoint extension of $T_0$ has a discrete spectrum.

Note that the proof above excludes the case $k=1$. For this case the conclusion of Lemma 1 becomes

$$|u(x_2) - u(x_1)| \leqq (T_0 u, u)^{1/2} \left( \int_{x_1}^{x_2} \frac{1}{p_1(t)}\, dt \right)^{1/2},$$

and the proof is given in the first step by setting $k=1$. The corresponding result of Lemma 2(a) becomes

$$|u(x_2) - u(x_1)| \leqq (Tu, u)^{1/2} \left( \int_{x_1}^{x_2} \frac{1}{p_1(t)}\, dt \right)^{1/2}$$

and the proofs of Lemma 2(a) and the Theorem follow the same pattern for $k=1$ as given above for $k>1$.

4. **Remarks**. The Theorem provides a test for the spectrum of $T$ to be discrete based on an analysis of $p_k(x)$, for $k$ some integer in $[1, n]$. For a test based on $p_0(x)$ see [**5**, Theorem 2, p. 210].

Eastham's technique [**3**] is based on a comparison to the Euler operator and requires $b=\infty$. However, the procedure used in this paper does not depend on comparison with known results, and the result includes the case of a singularity at a finite endpoint. If $p_1(x)=(1+x)^{2+\alpha}$, $m(x)=1$, $0 \leqq x < \infty$, then both results conclude a discrete spectrum if $\alpha>0$. Eastham obtains a discrete spectrum for

$$p_k(x) = (x + 2)^{2k} \log(x + 2), \qquad 0 \leqq x < \infty,$$

but no such conclusion can be drawn from our theorem. It is possible to construct examples of oscillatory $p_k(x)$ for which $\liminf p_k(x)/x^{2k}$ is finite and $\limsup p_k(x)/x^{2k}$ is infinite and, hence, Eastham's results are inconclusive, but for which our theorem concludes a discrete spectrum.

Our theorem, supplemented by Theorem 2, [**5**, p. 210], yields the results that Friedrichs obtained [**4**] for those examples for which a discrete spectrum exists. Note that, for the Laguerre operator, a change of variable is needed to obtain the familiar form

$$tu = -(xu')' + (x/4 - \tfrac{1}{2})u, \qquad 0 \leqq x < \infty,$$

in which case $p_0(x) = (x/4 - \frac{1}{2}) \to \infty$ as $x \to \infty$ and the discrete spectrum follows.

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Department of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332