

CRITERIA FOR DISCRETE SPECTRUM OF SINGULAR SELFADJOINT DIFFERENTIAL OPERATORS

L. W. ROLLINS

ABSTRACT. Under certain conditions on the coefficients of symmetric singular differential operators of order $2n$, selfadjoint extensions are shown to have a discrete spectrum. The results are proven specifically for the Friedrichs extension.

1. Introduction. Consider the formally selfadjoint differential operator defined by

$$\tau u(x) = \frac{1}{m(x)} \sum_{k=0}^n (-1)^k D^k p_k(x) D^k u(x), \quad a \leq x < b,$$

where b may be finite or infinite, p_k is real-valued, nonnegative, and in $C^{(k)}[a, b)$ for $0 \leq k \leq n$, and, for $x \in [a, b)$, $p_n(x) > 0$, and $m(x) > 0$. The endpoint b is possibly a singular endpoint.

Let $L^2(m)$ be the Hilbert space of all measurable functions, f , defined on (a, b) for which $\int_a^b |f(x)|^2 m(x) dx < \infty$, with inner product and norm denoted by (\cdot, \cdot) and $\|\cdot\|$ respectively. Let T_0 be the symmetric linear operator defined by $T_0 u = \tau u$ whose domain $D(T_0)$ consists of those functions in $L^2(m)$ which have $2n$ continuous derivatives and compact support in (a, b) . Then $D(T_0)$ is dense in $L^2(m)$. Under these conditions T_0 will have a selfadjoint extension; in particular, it will have the Friedrichs extension [6, p. 124].

This paper is concerned with finding conditions on the functions p_k , $1 \leq k \leq n$, which guarantee that the Friedrichs extension has a compact inverse and, hence, a discrete spectrum. Since all selfadjoint extensions of T_0 have the same continuous spectrum [5, Theorem 2, p. 92], we conclude that every selfadjoint extension of T_0 has a discrete spectrum.

Early investigations were confined to second order differential operators with the weight function $m(x) = 1$ and are available in the literature (see for example: Naimark [5, Chapter VII], and Dunford-Schwartz [2, Chapter XIII]). Recently Eastham [3] obtained results for higher order

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differential operators with $b = \infty$. The results here will be compared with Eastham's and, also, earlier results of Friedrichs [4]. The techniques are an extension of those used by Baxley [1].

2. Statement of theorem. The operator T_0 is symmetric and semi-bounded from below. To see the latter, use repeated integration by parts and the fact that $p_k \geq 0$ for $0 \leq k \leq n$, to get

$$(T_0 u, u) = \int_a^b \sum_{k=0}^n p_k(x) [u^{(k)}(x)]^2 dx \geq 0$$

for $u \in D(T_0)$. Therefore T_0 has a Friedrichs extension, T .

Let k be an integer, $1 \leq k \leq n$. For this k define a sequence of functions $\{h_i\}_1^k$ as follows:

If, for each y in $[a, b)$, it is true that $\int_y^b 1/p_k(t) dt$ is finite, then

$$(1a) \quad h_1(x, k) = \left(\int_x^b \frac{1}{p_k(t)} dt \right)^{1/2}, \quad \text{for } x \in [a, b).$$

Otherwise

$$(1b) \quad h_1(x, k) = \left(\int_a^x \frac{1}{p_k(t)} dt \right)^{1/2}, \quad \text{for } x \in [a, b).$$

If i is any integer $1 \leq i \leq k-1$, and if, for each y in $[a, b)$, it is true that $\int_y^b h_i(t, k) dt$ is finite, then

$$(2a) \quad h_{i+1}(x, k) = \int_x^b h_i(t, k) dt, \quad \text{for } x \in [a, b).$$

Otherwise

$$(2b) \quad h_{i+1}(x, k) = \int_a^x h_i(t, k) dt, \quad \text{for } x \in [a, b).$$

THEOREM. Let T be the Friedrichs extension of T_0 . If

$$(3) \quad \int_a^b m(x) [h_k(x, k)]^2 dx = M < \infty$$

for at least one integer k , $1 \leq k \leq n$, then T has a compact inverse, and in this case, every selfadjoint extension of T_0 has a discrete spectrum.

3. Proof of theorem. We first prove two lemmas.

LEMMA 1. If $u \in D(T_0)$ and (3) is satisfied for some k , $1 < k \leq n$, then for $a \leq x_1 < x_2 < b$,

$$|u(x_2) - u(x_1)| \leq (T_0 u, u)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t, k) dt.$$

PROOF. For $u \in D(T_0)$, it follows from the Schwartz inequality

$$\begin{aligned} |u^{(k-1)}(x_2) - u^{(k-1)}(x_1)|^2 &\leq \left| \int_{x_1}^{x_2} u^{(k)}(t) dt \right|^2 \\ &\leq \int_{x_1}^{x_2} p_k(t) [u^{(k)}(t)]^2 dt \int_{x_1}^{x_2} \frac{1}{p_k(t)} dt \\ &\leq (T_0 u, u) \int_{x_1}^{x_2} \frac{1}{p_k(t)} dt. \end{aligned}$$

Now setting $x_1 = x$ and $x_2 = b$, or $x_1 = a$ and $x_2 = x < b$ for $h_1(x, k)$ defined as in (1a) or (1b) respectively, leads to the following:

$$(4) \quad \begin{aligned} |u^{(k-1)}(x)|^2 &\leq (T_0 u, u) [h_1(x, k)]^2, \\ |u^{(k-1)}(x)| &\leq (T_0 u, u)^{1/2} h_1(x, k). \end{aligned}$$

Using (4) above, we obtain

$$\begin{aligned} |u^{(k-2)}(x_2) - u^{(k-2)}(x_1)| &= \left| \int_{x_1}^{x_2} u^{(k-1)}(t) dt \right| \\ &\leq \left(\int_{x_1}^{x_2} (T_0 u, u)^{1/2} h_1(t, k) dt \right) = (T_0 u, u)^{1/2} \left(\int_{x_1}^{x_2} h_1(t, k) dt \right). \end{aligned}$$

Setting $x_1 = x$ and $x_2 = b$, or $x_1 = a$ and $x_2 = x < b$ for $h_2(x, k)$ defined as in (2a) or (2b) respectively,

$$\begin{aligned} |u^{(k-2)}(x)|^2 &\leq (T_0 u, u) [h_2(x, k)]^2, \\ |u^{(k-2)}(x)| &\leq (T_0 u, u)^{1/2} h_2(x, k). \end{aligned}$$

Continuing by induction, we get the inequality

$$|u^{(k-j)}(x)| \leq (T_0 u, u)^{1/2} h_j(x, k), \quad j = 1, \dots, k-1,$$

in particular, for $j = k-1$,

$$|u'(x)| \leq (T_0 u, u)^{1/2} h_{k-1}(x, k).$$

Integration of both sides leads to the conclusion of the lemma.

LEMMA 2. Let T be the Friedrichs extension of T_0 . If $u \in D(T)$ and (3) holds for some k , $1 < k \leq n$, then for $a \leq x_1 < x_2 < b$,

$$(a) \quad |u(x_2) - u(x_1)| \leq (Tu, u)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t, k) dt,$$

$$(b) \quad (u, u) \leq (Tu, u)M.$$

PROOF. For $u \in D(T)$ there exists a sequence $\{u_m\}$ with $u_m \in D(T_0)$, $m=1, \dots$, such that $\|u_m - u\| \rightarrow 0$ and $(T_0 u_m, u_m) \rightarrow (Tu, u)$ as $m \rightarrow \infty$.

Since $(T_0 u_m, u_m)$ is a convergent sequence it is bounded, that is $(T_0 u_m, u_m) \leq C$ for some positive number C , $m=1, \dots$. From Lemma 1,

$$(5) \quad |u_m(x_2) - u_m(x_1)| \leq (T_0 u_m, u_m)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t, k) dt.$$

Therefore the functions $\{u_m\}$ are equicontinuous and uniformly bounded on compact subsets of $[a, b]$. By the Ascoli Theorem there is a subsequence of $\{u_m\}$ which converges uniformly on compact subsets of $[a, b]$. Restricting attention to this subsequence and taking limits in (5) we get (a),

$$|u(x_2) - u(x_1)| \leq (Tu, u)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t, k) dt.$$

For the case $h_k(x, k) = \int_a^x h_{k-1}(t, k) dt$, let $x_1 = a$, $x_2 = x$ in the above inequality to obtain

$$|u(x)| \leq (Tu, u)^{1/2} h_k(x, k).$$

For the case $h_k(x, k) = \int_x^b h_{k-1}(t, k) dt$, then note that for each u_m in the subsequence there is a $x_2 < b$ such that $u_m(x_2) = 0$. Setting $x_1 = x$ we obtain

$$\begin{aligned} |u_m(x)| &= |u_m(x_2) - u_m(x)| \\ &\leq (T_0 u_m, u_m)^{1/2} \int_x^{x_2} h_{k-1}(t, k) dt \leq (T_0 u_m, u_m)^{1/2} \int_x^b h_{k-1}(t, k) dt \\ &= (T_0 u_m, u_m)^{1/2} h_k(x, k). \end{aligned}$$

Now take limits to obtain, for this case also, $|u(x)| \leq (Tu, u)^{1/2} h_k(x, k)$. Hence, in either case

$$(6) \quad |u(x)|^2 \leq (Tu, u)[h_k(x, k)]^2.$$

Integration of the above leads to (b).

PROOF OF THE THEOREM. Let λ be an eigenvalue of T associated with eigenvector u . Then $(Tu, u) = (\lambda u, u) \geq 1/M(u, u)$. Hence, $\lambda \geq 1/M > 0$. Therefore all eigenvalues of T are positive and T^{-1} exists.

Let $\{Tu_n\}$ be a bounded sequence in the domain of T^{-1} and K be a number such that $\|Tu_n\| \leq K < \infty$, $n=1, \dots$. Then $u_n \in D(T)$ and, from Lemma 2(b),

$$\|u_n\|^2 = (u_n, u_n) \leq M(Tu_n, u_n) \leq M \|Tu_n\| \|u_n\|,$$

from which follows $\|u_n\| \leq MK$, and $(Tu_n, u_n) \leq MK^2$.

From Lemma 2(a),

$$|u_n(x_2) - u_n(x_1)| \leq (MK^2)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t, k) dt$$

and, hence, $\{u_n\}$ is uniformly bounded and equicontinuous on compact subsets of $[a, b)$. Using the Ascoli Theorem we get a subsequence $\{v_n\}$ which converges uniformly on compact subsets of $[a, b)$ to a limit function u .

It remains to show that $\{v_n\}$ converges in $L^2(m)$. From (6) above

$$\begin{aligned} m(x) |v_n(x)|^2 &\leq (Tv_n, v_n)m(x)[h_k(x, k)]^2 \\ &\leq MK^2m(x)[h_k(x, k)]^2, \quad n = 1, \dots \end{aligned}$$

The right side is integrable by hypothesis. The Lebesgue Dominated Convergence Theorem yields $u \in L^2(m)$ and $\|v_n\| \rightarrow \|u\|$. Hence,

$$\|v_n - u\| \rightarrow 0.$$

Hence, T^{-1} is compact, and T has a discrete spectrum. It follows [5, Theorem 2, p. 92] that every selfadjoint extension of T_0 has a discrete spectrum.

Note that the proof above excludes the case $k=1$. For this case the conclusion of Lemma 1 becomes

$$|u(x_2) - u(x_1)| \leq (T_0u, u)^{1/2} \left(\int_{x_1}^{x_2} \frac{1}{p_1(t)} dt \right)^{1/2},$$

and the proof is given in the first step by setting $k=1$. The corresponding result of Lemma 2(a) becomes

$$|u(x_2) - u(x_1)| \leq (Tu, u)^{1/2} \left(\int_{x_1}^{x_2} \frac{1}{p_1(t)} dt \right)^{1/2}$$

and the proofs of Lemma 2(a) and the Theorem follow the same pattern for $k=1$ as given above for $k>1$.

4. Remarks. The Theorem provides a test for the spectrum of T to be discrete based on an analysis of $p_k(x)$, for k some integer in $[1, n]$. For a test based on $p_0(x)$ see [5, Theorem 2, p. 210].

Eastham's technique [3] is based on a comparison to the Euler operator and requires $b=\infty$. However, the procedure used in this paper does not depend on comparison with known results, and the result includes the case of a singularity at a finite endpoint. If $p_1(x)=(1+x)^{2+\alpha}$, $m(x)=1$, $0 \leq x < \infty$, then both results conclude a discrete spectrum if $\alpha > 0$. Eastham obtains a discrete spectrum for

$$p_k(x) = (x + 2)^{2k} \log(x + 2), \quad 0 \leq x < \infty,$$

but no such conclusion can be drawn from our theorem. It is possible to construct examples of oscillatory $p_k(x)$ for which $\liminf p_k(x)/x^{2k}$ is finite and $\limsup p_k(x)/x^{2k}$ is infinite and, hence, Eastham's results are inconclusive, but for which our theorem concludes a discrete spectrum.

Our theorem, supplemented by Theorem 2, [5, p. 210], yields the results that Friedrichs obtained [4] for those examples for which a discrete spectrum exists. Note that, for the Laguerre operator, a change of variable is needed to obtain the familiar form

$$tu = -(xu')' + (x/4 - \frac{1}{2})u, \quad 0 \leq x < \infty,$$

in which case $p_0(x) = (x/4 - \frac{1}{2}) \rightarrow \infty$ as $x \rightarrow \infty$ and the discrete spectrum follows.

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DEPARTMENT OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GEORGIA 30332