PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, July 1972

FIXED POINTS BY MEAN VALUE ITERATIONS

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ABSTRACT. If E is a convex compact subset of a Hilbert space, T is a strictly pseudocontractive function from E into E and x_1 is a point in E, then the point sequence $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of T, where for each positive integer n,

$$x_{n+1} = [1/(n + 1)][Tx_n + nx_n].$$

In this paper it is shown that a technique of W. R. Mann [1] is fruitful in finding a fixed point of a function even though a "Picard" iteration may fail. The technique of Mann is similar to some of the techniques used in [2].

THEOREM. *If E is a compact convex subset of a Hilbert space H, T is a strictly pseudocontractive function from E into E and x_1 is any point in E, then the point sequence $\{x_i\}_{i=1}^{\infty}$ converges in the norm of H to a fixed point of T, where for each positive integer n, $x_{n+1}=[1/(n+1)][Tx_n+nx_n]$.*

PROOF. By a well-known result of Schauder [3], T has a fixed point in E. Let p denote one such fixed point of T and x_1 a point in E. Since H is a Hilbert space, it is easily shown that for each positive integer n,

$$\begin{aligned}\|x_{n+1} - p\|^2 = {} & [1/(n + 1)]\ \|Tx_n - p\|^2 \\ & + [n/(n + 1)]\ \|x_n - p\|^2 - [n/(n + 1)^2]\ \|Tx_n - x_n\|^2\end{aligned}$$

where $x_{n+1}=[1/(n+1)][Tx_n+nx_n]$.

Since T is strictly pseudocontractive, there is a number k, $0\leqq k<1$, such that if each of x and y is in E then

$$\|Tx - Ty\|^2 \leqq \|x - y\|^2 + k\ \|(I - T)x - (I - T)y\|^2.$$

Hence

$$\|Tx_n - p\|^2 \leqq \|x_n - p\|^2 + k\ \|x_n - Tx_n\|^2,$$

$$\|x_{n+1} - p\|^2 \leqq \|x_n - p\|^2 - [[n - k(n + 1)]/(n + 1)^2]\ \|Tx_n - x_n\|^2.$$

Received by the editors November 22, 1971.

AMS 1970 *subject classifications.* Primary 47A85; Secondary 40A32.

There is a positive integer N such that $N/(N+1)>k$ and therefore if j is a positive integer

$$\|x_{N+j}-p\|^2 \leqq \|x_{N+j-1}-p\|^2 - [[(1-k)(N+j-1)-k]/(N+j)^2]\,\|Tx_{N+j-1}-x_{N+j-1}\|^2$$

from which it follows that

$$\|x_{N+j+1}-p\|^2 \leqq \|x_N-p\|^2 - \sum_{t=0}^{j} [[(1-k)(N+t)-k]/(N+t+1)^2]\,\|Tx_{N+t}-x_{N+t}\|^2.$$

Since E is bounded the above implies that

$$\sum_{t=0}^{\infty} [[(1-k)(N+t)-k]/(N+t+1)^2]\,\|Tx_{N+t}-x_{N+t}\|^2 < \infty$$

and $\|x_{N+j+1}-p\| \leqq \|x_{N+j}-p\|$ for $j=1, 2, \cdots$.

Hence

$$\lim_{t\to\infty} \|Tx_{N+t}-x_{N+t}\| = 0.$$

Since E is compact there is a subsequence $\{x_{N+n_i}\}_{i=1}^{\infty}$ which converges to a point q in E. Notice that q is a fixed point of T.

Since q is a fixed point of T we have for each positive integer j that

$$\|x_{N+j+1}-q\| \leqq \|x_{N+j}-q\|.$$

This last comment combined with the fact that $\{x_{N+n_i}\}_{i=1}^{\infty}$ converges to q yields the desired result, namely that $\{x_i\}_{i=1}^{\infty}$ converges to q, which is a fixed point of T.

References

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