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FINITE UNIONS OF CONVEX SETS[1]

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ABSTRACT. In this paper it is shown that a set is the union of k convex subsets if and only if every finite subset of it is contained in some k convex subsets of it. This is a characterization of a set as the union of a finite number of convex sets by conditions on its finite subsets.

Also, a proof of McKinney's theorem for unions of two convex sets is given using similar methods.

Richard McKinney has given a characterization of unions of two convex sets. In this paper a complete characterization of unions of convex sets is given, and we give another proof of McKinney's result.

Here, a *k-partition* of a set S is a family $\mathscr{S}=\{S_1, S_2, \cdots, S_k\}$ of subsets of S, having k elements, where $S_i \cap S_j=\varnothing$ for $i\neq j$, and $\bigcup_{i=1}^k S_i=S$.

A property P of sets is said to be *hereditary* if, given any set S with property P, any subset $T\subset S$ has property P.

I. First, we prove a theorem which enables us to by-pass further reference to Zorn's Lemma.

THEOREM 1. *Let P be a hereditary property of sets. Let S be a set such that for every finite subset $F\subset S$, there is a k-partition of F, $\{F_1, F_2, \cdots, F_k\}$, such that F_i has property P, $1\leqq i\leqq k$. Then there is a k-partition of S, $\{S_1, S_2, \cdots, S_k\}$, such that for every finite subset $F\subset S$, $F\cap S_i$ has property P, $1\leqq i\leqq k$.*

PROOF. Let $\mathscr{F}=\{\alpha\subset S|\alpha$ is finite$\}$; and, for each $\alpha\in\mathscr{F}$, let $P_\alpha=\{(F_1, \cdots, F_k)|F_i$ has property P, $1\leqq i\leqq k$, and $\{F_1, \cdots, F_k\}$ is a k-partition of $\alpha\}$. Since α is finite, P_α is finite and, with the discrete topology, P_α is a compact topological space. The Tychonoff theorem gives $\prod_{\alpha\in\mathscr{F}} P_\alpha$ to be compact. (For $X\in\prod_{\alpha\in\mathscr{F}} P_\alpha$, denote the αth coordinate of X by X_α.

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Also, X_α is a k-tuple of subsets of S and the ith is denoted $X_{\alpha i}$, $1 \leqq i \leqq k$.)

For $\alpha \in \mathscr{F}$ define

$$A_\alpha = \left\{ X \in \prod_{\alpha \in \mathscr{F}} P_\alpha \middle| \text{ if } \beta \subset \alpha, \text{ then } X_{\beta i} = X_{\alpha i} \cap \beta,\ 1 \leqq i \leqq k \right\}.$$

It is easy to verify that A_α is closed for each $\alpha \in \mathscr{F}$ and it is now shown that, likewise, each $A_\alpha \neq \varnothing$. Since $P_\beta \neq \varnothing$, by hypothesis, select a $Q_\beta \in P_\beta$ for each $\beta \in \mathscr{F}$. Then, for $\alpha \in \mathscr{F}$ let

$$\begin{aligned} R^\alpha_{\beta i} &= Q_{\beta i} && \text{if } \beta \not\subset \alpha, \\ &= Q_{\alpha i} \cap \alpha && \text{if } \beta \subset \alpha; \end{aligned}$$

then it is immediate that the point R^α thus defined is in A_α. Next, observe that

$$\varnothing \neq A_{\alpha_1 \cup \cdots \cup \alpha_n} \subset A_{\alpha_1} \cap \cdots \cap A_{\alpha_n}$$

for any positive integer n. Thus, the family $\{A_\alpha\}_{\alpha \in \mathscr{F}}$ is a collection of non-empty, closed subsets of the compact space $\prod_{\alpha \in \mathscr{F}} P_\alpha$ having the finite intersection property. It follows that there exists an $X \in \bigcap_{\alpha \in \mathscr{F}} A_\alpha$.

Now define $S_i = \{x \in S \mid \{x\} = X_{\{x\}i}\}$. It follows easily that for each $\alpha \in \mathscr{F}$, $S_i \cap \alpha = X_{\alpha i}$. It is also routine to see that $\{S_1, \cdots, S_k\}$ partitions S. Since $S_i \cap \alpha = X_{\alpha i}$, it follows that $S_i \cap \alpha$ has property P, so $\{S_1, \cdots, S_k\}$ is the desired partition of S.

II. THEOREM 2. *Let S be a subset of a linear space such that each finite subset $F \subset S$ has a k-partition, $\{F_1, F_2, \cdots, F_k\}$, where* conv $F_i \subset S$, *$1 \leqq i \leqq k$. Then S is the union of k convex sets.*

PROOF. Let a set $T \subset S$ have property P if and only if conv $T \subset S$. This property is hereditary, and any finite subset $F \subset S$ has a k-partition, $\{F_1, F_2, \cdots, F_k\}$, such that each F_i has property P. By Theorem 1, S has a k-partition, $\{S_1, S_2, \cdots, S_k\}$, such that for any finite subset $F \subset S$, $F \cap S_i$ has property P, $1 \leqq i \leqq k$. In particular, if F is a finite subset of S_i, conv $F \subset S$. This implies that conv $S_i \subset S$, since

$$\operatorname{conv} S_i = \bigcup \{\operatorname{conv} F \mid F \text{ is a finite subset of } S_i\}.$$

We have

$$S = \bigcup_{i=1}^{k} S_i = \bigcup_{i=1}^{k} (\operatorname{conv} S_i).$$

We prove the following statement of McKinney's theorem:

THEOREM 3. *Let S be a closed subset of a topological linear space such that for every finite subset $F \subset S$, there is a 2-partition $\{F_1, F_2\}$ such that if $x, y \in F_i$ $(1 \leqq i \leqq 2)$ then $xy \subset S$. Then S is the union of two convex sets.*

PROOF. Let the set $T\subset S$ have property P if and only if, for any x, $y\in T$, $xy\subset S$. This property is hereditary, and any finite subset $F\subset S$ has a 2-partition $\{F_1, F_2\}$ such that F_1 and F_2 have property P. By Theorem 1, S has a 2-partition, $\{S_1, S_2\}$, such that for any finite subset $F\subset S$, $F\cap S_1$ and $F\cap S_2$ have property P. Clearly, $S_1\cup S_2=S$; for $x, y\in S_1$, $xy\subset S$; and for $x, y\in S_2$, $xy\subset S$.

For $x\in S$, let $\operatorname{st} x=\{y\in S \mid xy\subset S\}$. Let $\{y_i\}_{i=1}^{\infty}$ be a sequence in $\operatorname{st} x$ with limit y_0; $y_0\in S$ since S is closed. For any element $\lambda x+(1-\lambda)y_0$ $(0\leqq\lambda\leqq 1)$ of xy_0, the sequence $\{\lambda x+(1-\lambda)y_i\}_{i=1}^{\infty}$ is in S and has limit $\lambda x+(1-\lambda)y_0$, so $\lambda x+(1-\lambda)y_0$ is in S. Therefore $xy_0\subset S$, $y_0\in \operatorname{st} x$, and $\operatorname{st} x$ is closed.

Let $A=\bigcap_{x\in S_1}(\operatorname{st} x)$ and $B=\bigcap_{x\in S_2}(\operatorname{st} x)$; $S_1\subset A$ and $S_2\subset B$, so $A\cup B=S_1\cup S_2=S$. We want to show that A and B are convex.

Clearly, A and B are closed sets, and $A\cap B$ is the convex kernel of S.

Let a_1 and a_2 be elements of A. Either both a_1 and a_2 are in S_2, or at least one is in S_1 and the other is in $A=\bigcap_{x\in S_1}(\operatorname{st} x)$. In either case, $a_1a_2\subset S$. It is necessary to show that $a_1a_2\subset A$.

Let c be an element of a_1a_2 other than a_1 and a_2. If $c\notin A$ then $c\in B$. Since A is closed, c is in an open-ended segment of a_1a_2, say (b_1b_2), such that $(b_1b_2)\subset S-A$ and b_1, $b_2\in A$. But $(b_1b_2)\subset S-A\subset B$, and since B is closed b_1, $b_2\in B$. We have b_1, $b_2\in A\cap B$, the convex kernel of S. But this means $c\in A\cap B\subset A$.

Therefore, $a_1a_2\subset A$ for a_1, $a_2\in A$, so A is convex. Similarly, B is convex, and $S=A\cup B$.

III. Theorem 1 also gives an immediate proof of the next theorem. The easy details are omitted.

THEOREM 4. *Let $G=(V, E)$ be a graph, possibly infinite, with no multiple edges or loops. Then G has chromatic number k if and only if every finite induced subgraph of G has chromatic number at most k, and there is at least one such subgraph whose chromatic number is precisely k.*

Various efforts (see **[1]**, **[2]**, [3], **[4]**) have been made to characterize finite unions of convex sets by simple conditions on finite subsets of points in the resulting set. The McKinney theorem is the first such characterization for $k=2$. In all these prior efforts, however, the basic appoach has been through conditions about segments (convex hulls of pairs of points) rather than convex hulls of larger finite sets. The present approach eventually presents itself as a very "natural" method and yields a reasonable characterization.

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