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SEMIRINGS OF FUNCTIONS DETERMINE FINITE T_0 TOPOLOGIES

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ABSTRACT. An analogue of the Stone-Gelfand-Kolmogoroff theorem for compact Hausdorff spaces is proven for finite T_0 topological spaces. Let $C(X)$ be the semiring of continuous functions from finite T_0 X into Z, the nonnegative integers with open sets of the form $\{0, 1, 2, \cdots, m\}$. Products and sums in $C(X)$ are defined pointwise. Denote the set of nonzero semiring homomorphisms of $C(X)$ into Z by $H(X)$ and give it the compact-open topology where $C(X)$ is considered discrete. Then (1) X and $H(X)$ are homeomorphic. (2) $C(X)$ is semiring isomorphic to $C(Y)$ iff X is homeomorphic to Y. (3) The topology of X can be completely recovered from the inclusion relations among the ideals of $C(X)$ which are kernels of the elements in $H(X)$.

The study of finite topological spaces may at first seem somewhat contrived. However finite spaces do naturally occur and they can be useful for something other than nice counterexamples. For instance, recently in [1] they were used in a metrization problem. Since computers (and perhaps even the entire universe) are finite, some finite spaces may have application to practical physical problems.

A natural question is the extent to which maps on a space determine its topology. For example, the classical result for compact Hausdorff spaces is that the topology is determined by the algebraic structure of the ring of real valued continuous functions. In this paper it is shown that the semiring $C(X)$ of continuous functions from a finite T_0 space X into the nonnegative integers Z with an order induced A-space topology characterizes the topology of X. The method of proof is to first show X is homeomorphic to the set $H(X)$ of nontrivial semiring homomorphisms from $C(X)$ to Z. Here $C(X)$ is considered discrete and $H(X)$ has the compact open topology. It then follows that $C(X)$ and $C(Y)$ are semiring isomorphic if and only if X and Y are homeomorphic.

The following facts about finite spaces and A-spaces will be used. Good references are [3] and [4]. A partial order on a set X determines a topology on X by defining the minimal open set containing i to be those j with $j \leqq i$.

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X is T_0 iff $i<j$ and $j<i$ implies $i=j$. Any finite T_0 topology can be determined in this manner. A function $f: X\to Y$, with the topologies of X and Y defined by partial orders, is continuous iff $i<j$ in X implies $f(i)\leqq f(j)$ in Y. Hereafter $X=\{1, 2, \cdots, n\}$ has some fixed partial order such that X is a finite T_0 space. Let $Z=\{0, 1, 2, \cdots, n, n+1, \cdots\}$. Using the natural order as a partial order the topology has open sets $\varnothing$, Z, and all sets of the form $\{0, 1, \cdots, m\}$.

Let $C(X)$ denote the set of continuous functions from X into Z. Then a typical element $t\in C(X)$ will be viewed as an ordered n-tuple $t=(t_1, t_2, \cdots, t_n)$ of nonnegative integers where $t(i)=t_i$. Thus $C(X)$ is the set of ordered n-tuples with $i<j$ in X implying $t_i\leqq t_j$. $C(X)$ becomes a semiring by defining sums and products componentwise. Thus $(t+s)_i=t_i+s_i$ and $(ts)_i=t_is_i$. It is easy to check that sums and products of continuous functions are again continuous. $C(X)$ has a partial order given by $t<s$ iff $t_i\leqq s_i$ for all i. The compact-open topology on $C(X)$ is the same as the topology determined by this order. Under this topology addition and multiplication are continuous.

The following result shows that if the elements of $C(X)$ are given as ordered n-tuples, the topology of X can be recovered from the set $C(X)$. In fact, only continuous functions into the subspace $\{0, 1\}$ need be considered.

Lemma 1. *In X, $i<j$ iff $t_i\leqq t_j$ for all $t\in C(X)$.*

Proof. By continuity, for any t, $i<j$ implies $t_i\leqq t_j$. Conversely, suppose $i\not< j$. Define $t: X\to Z$ by $t(l)=1$ for $i\leqq l$ and $t(l)=0$ otherwise. Then $t^{-1}(\{0, 1, \cdots, r\})=X$ for $r\geqq 1$. Suppose $h\in t^{-1}(0)$ and $k<h$ in X. If $t(k)=1$ then $i\leqq k<h$ which contradicts $h\in t^{-1}(0)$. Thus $t(k)=0$, $k\in t^{-1}(0)$ and $t^{-1}(0)$ is an open set. Hence t is continuous. But $t_i=1$ and $t_j=0$ so $t_i\not\leqq t_j$ for all $t\in C(X)$.

Define $H(X)$ to be the set of nonzero semiring homomorphisms from $C(X)$ into Z where addition and multiplication in Z are the usual ones. The next two results are needed to study $H(X)$.

Theorem 2. *Let C be a set of ordered n-tuples of nonnegative integers closed under coordinatewise addition and multiplication. Assume C contains a nonzero element of the form $(a, a, \cdots, a)$. Then $f: C\to Z$ preserves sums and products iff f is the projection on some fixed coordinate.*

Proof. The projection p_j onto the jth coordinate clearly preserves sums and products. Suppose $f: C\to Z$ also preserves both operations and $f(t_1, t_2, \cdots, t_n)=r$. Consider the polynomial equation $P(x)=(x-t_1)(x-t_2)\cdots(x-t_n)=0$. If n is even, define $Q(x)$ to be the sum of the even degree terms in $P(x)$ and $R(x)$ to be minus the sum of the odd degree terms. If n is odd, interchange the roles of $Q(x)$ and $R(x)$. In either case the coefficients of $Q(x)$ and $R(x)$ are all positive and $P(x)=0$ iff $Q(x)=R(x)$. Since f preserves products $f(t_1^q, t_2^q, \cdots, t_n^q)=r^q$. Since f also preserves sums, $f(Q(t_1), Q(t_2), \cdots, Q(t_n))=Q(r)$ and $f(R(t_1), R(t_2), \cdots, R(t_n))=R(r)$. But since $Q(t_i)=R(t_i)$ for $i=1, \cdots, n$ it follows that $Q(r)=R(r)$. Thus $P(r)=0$ or $r=t_j$ for some j.

Now suppose f is not projection on any coordinate. Then for each $i=1, 2, \cdots, n$ there exists an element $\boldsymbol{i}=(\boldsymbol{i}_1, \boldsymbol{i}_2, \cdots, \boldsymbol{i}_n)$ in C such that $f(\boldsymbol{i})=r_i$ with $\boldsymbol{i}_i\neq r_i$. By adding $(a, a, \cdots, a)$ if necessary, we can assume $r_i\neq 0$ and $\boldsymbol{i}_j\neq 0$ for all i and j. Since products are preserved, $f(\mathbf{1}^{a_1}\mathbf{2}^{a_2}\cdots\boldsymbol{n}^{a_n})=r_1^{a_1}r_2^{a_2}\cdots r_n^{a_n}$ for all positive exponents $a_1, a_2, \cdots, a_n$. For fixed $a_1, a_2, \cdots, \hat{a}_i, \cdots, a_n$ there is at most one value of a_i for which $\mathbf{1}_i^{a_1}\mathbf{2}_i^{a_2}\cdots\boldsymbol{i}_i^{a_i}\cdots\boldsymbol{n}_i^{a_n}=r_1^{a_1}r_2^{a_2}\cdots r_i^{a_i}\cdots r_n^{a_n}$. If there were two such values it would follow that $\boldsymbol{i}_i=r_i$ which is a contradiction. Now consider the set of $\{a_1, a_2, \cdots, a_n\}$ with $1\leqq a_j\leqq n+1$. This set has $(n+1)^n$ elements.

For a fixed i there are $(n+1)^{n-1}$ possible choices of $a_1, a_2, \cdots, \hat{a}_i, \cdots, a_n$ and hence only $(n+1)^{n-1}$ possible times that $r_1^{a_1}r_2^{a_2}\cdots r_n^{a_n}$ could equal $\mathbf{1}_i^{a_1}\cdots\boldsymbol{i}_i^{a_i}\cdots\boldsymbol{n}_n^{a_n}$. Thus there are only $n\cdot(n+1)^{n-1}$ possible times $r_1^{a_1}\cdots r_n^{a_n}$ could equal a coordinate of $\mathbf{1}^{a_1}\cdots\boldsymbol{n}^{a_n}$. But $n\cdot(n+1)^{n-1}<(n+1)^n$ which contradicts the fact that $f(t_1, \cdots, t_n)$ is some t_j. Therefore f must be a projection on some fixed factor.

Lemma 3. *Let $C(X)$ have the discrete topology. Then the compact open topology on $H(X)$ is the same as the topology defined by the partial order $f<g$ iff $f(t)\leqq g(t)$ for all $t\in C(X)$.*

Proof. By Theorem 2, the elements of $H(X)$ are exactly the projections onto the coordinates, p_i, $i=1, \cdots, n$. By Lemma 1, $p_i<p_j$ iff $i<j$. Let $V=\{p_j | j\leqq i\}$ be a minimal open set in the partial order topology. Define $t:X\to Z$ by $t(l)=0$ if $l\leqq i$ and $t(l)=1$ otherwise. Clearly $t\in C(X)$. Then $U=\{p_k | p_k(t)\in\{0\}\}$ is open in the compact open topology and $U=V$. Conversely consider $U=\{p_k | p_k(s)\in\{0, 1, \cdots, r\}\}$ a subbasic open set in the compact open topology. Let $p_k\in U$ and suppose $p_j\leqq p_k$. Then $p_j(s)\leqq p_k(s)$ so $p_j(s)\in\{0, 1, \cdots, r\}$ hence $p_j\in U$ and U is open in the partial order topology.

Theorem 4. *$H(X)$ with the compact open topology is homeomorphic to X.*

Proof. Define $\phi:X\to H(X)$ by $\phi(i)=p_i$, projection onto the ith coordinate. Suppose $\phi(i)=\phi(j)$. Then $t_i=t_j$ for all $t\in C(X)$. By Lemma 1 this means $i<j$ and $j<i$. Since X is assumed to be T_0, $i=j$ and ϕ is one-to-one. By Theorem 2, ϕ is onto. To show ϕ is continuous suppose $i<j$ in X.

Then $t_i \leqq t_j$ for all $t \in C(X)$. But this means $p_i \leqq p_j$ in $H(X)$. By Lemma 3 the topology on $H(X)$ is determined by the order. Thus $i < j$ implies $\phi(i) \leqq \phi(j)$ so ϕ is continuous. Conversely $\phi(i) \leqq \phi(j)$ implies $t_i \leqq t_j$ implies $i < j$ so ϕ^{-1} is continuous and ϕ is a homeomorphism.

With this result we can immediately obtain the next theorem. The proof is exactly that of [**2**, p. 290].

THEOREM 5. *Let X and Y be finite T_0 spaces and $h: C(Y) \rightarrow C(X)$ a semiring homomorphism. Then there exists a unique continuous function $\lambda: X \rightarrow Y$ which induces h by $h(t_1, t_2, \cdots, t_n) = (t_{\lambda(1)}, t_{\lambda(2)}, \cdots, t_{\lambda(n)})$. If h is an isomorphism, λ is a homeomorphism.*

The final theorem shows that the topology of X can be recovered from inclusion relations among some of the ideals in the semiring $C(X)$. Note that the ideals which work are not maximal ideals as one might expect.

THEOREM 6. *Let M_i be the kernel of $p_i \in H(X)$. Then $i < j$ in X iff $M_i \supset M_j$.*

PROOF. If $i < j$ then $p_i < p_j$. Suppose $t \in M_j \subset C(X)$. Then $p_i(t) \leqq p_j(t) = 0$ so $t \in M_i$. Conversely suppose $i \nleqq j$. Then define $s \in C(X)$ by $s(l) = 1$ if $i \leqq l$ and $s(l) = 0$ otherwise. Then $s(j) = 0$ and $s(i) = 1$ so $M_i \not\supset M_j$.

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