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FIBERED KNOTS THROUGH T-SURGERY

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ABSTRACT. Noninvertible fibered knots are constructed by a discrete surgical procedure called T-surgery on links.

A particular surgical technique will transform certain two-component links into knots. The process is called surgical transmutation or, more simply, T-surgery; the links used for this surgery are called admissible, and a knot obtained from an admissible link L is a *transmute* of L. The immediate importance of T-surgery is that one can produce knots with desired properties by its application to appropriately chosen admissible links; for example, the method yields a veritable cornucopia of noninvertible knots ([**11**], [**12**]).

The basic objective of this paper is to prove the existence of noninvertible *fibered* knots by T-surgery. These knots stand in contrast to H. F. Trotter's noninvertible, nonfibered pretzel knots [**9**], and to such knots as 8_{17} that are easily seen to be fibered (by the Murasugi test [**4**, Theorem 1.2, p. 544], say) but whose intuitively obvious noninvertibility feature has not yet been proved [**2**, Problem 10, p. 169].

In §1, we describe our surgical method, and give necessary and sufficient conditions (Theorem 1) for a transmute to be a prime knot. Our principal result (Theorem 2 of §2) states geometric conditions for a transmute to fiber. Intuitively Theorem 2 and the examples of §3 indicate an abundant variety of noninvertible fibered knots.

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1. **T-surgery and prime transmutes.** An oriented, ordered link $L=K_1\cup K_2$ of two components tamely imbedded in the oriented three-sphere $\mathfrak{S}$ is *admissible* if K_1 is of trivial knot type and L is unsplittable. Let L be an admissible link, denote by W the closure of the complement in $\mathfrak{S}$ of a tubular neighborhood of K_1 in $\mathfrak{S}-K_2$, and let V denote a *knotted*,

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closed solid torus, tamely imbedded in $\mathfrak{S}$. The (oriented) image $\mathscr{K}$ of K_2 under a faithful homeomorphism [7] of W onto V is a *transmute of L with respect to V*, and we shall say that $\mathscr{K}$ was obtained from L by *surgical transmutation* (or *T-surgery*).

THEOREM 1. *Let $L=K_1 \cup K_2$ be an admissible link, and let $\mathscr{K}$ denote a knot obtained by T-surgery on L. In order that $\mathscr{K}$ be a prime knot, it is both necessary and sufficient that L be a prime link and that the order* [7] *of W with respect to K_2 be $\geqq 2$.*

A proof of Theorem 1 may be constructed by straightforward applications of the definition of T-surgery and several theorems of H. Schubert [7]; we shall omit the details. Evidently, the transmutes of the link of Figure 1 in each of [**11**], [**12**], and the present paper are prime knots.

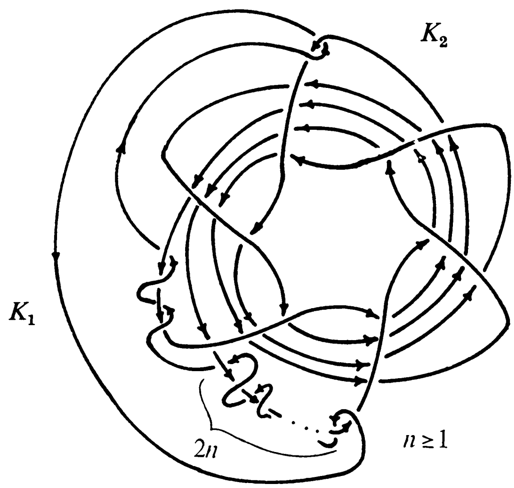

FIGURE 1

Theorem 1 can, with little trouble, also be applied to the links $\mathfrak{L}_{(+)}$ and $\mathfrak{L}_{(-)}$ (Figures 4 and 6, respectively of [**12**]).

2. **Fibered transmutes.** One should perhaps consider an example (Figure 1 of [**11**], [**12**], or the present paper) while reading

THEOREM 2. *$L=K_1 \cup K_2$ is an admissible link, and V_1 and V_2 represent disjoint, tubular neighborhoods of K_1 and K_2, respectively. Set $W=\mathfrak{S}-$* Int *V_1, let V denote a knotted, tame, closed solid torus in $\mathfrak{S}$, and suppose K is an* (*oriented*) *core of V. We assume that L has the following additional properties*:

(a) *K_2 is a fibered knot*;

(b) *the order of W with respect to K_2 coincides with the winding number w (see* [7]) *of K_2 in W;*

(c) *there is an orientable surface S^* of minimal genus properly imbedded in $\mathfrak{S}-\operatorname{Int} V_2$ and spanning a longitude of V_2, whose intersection with ∂W consists of exactly w longitudes of W.*

Denote by M^3 the space $W-\operatorname{Int} V_2$ split along the surface $S=S^*-\operatorname{Int} V_1$ (see [5]). *Besides the surfaces S_1 and S_2 that identify to yield S, and an annulus A that closes to become ∂V_2, there are w (closed) cylinders $\mathscr{C}_1, \cdots, \mathscr{C}_w$ such that*

$$\partial M^3 = S_1 \cup S_2 \cup A \cup \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_w.$$

Finally, suppose the following two conditions are satisfied:

(d) *K is a fibered knot;*

(e) *there exist w pairwise disjoint, polyhedral disks $\mathscr{D}_1, \cdots, \mathscr{D}_w$ properly imbedded in M^3 such that $\partial\mathscr{D}_i=\alpha_i\cup\beta_i$, where α_i is a properly imbedded arc of $\mathscr{C}_i$ connecting the two components of $\partial\mathscr{C}_i$, and where β_i is an arc properly imbedded in $S_1\cup S_2\cup A$ $(i=1, \cdots, w)$.*

Our conclusion is that the transmute $\mathscr{K}$ of L with respect to V is a fibered knot.

Proof. According to [7, Theorem, p. 192], our hypothesis (c) implies that the genus of $\mathscr{K}$ is $wg(K)+g(K_2)$, where $g(k)$ is the genus of a knot k. Let f be a faithful homeomorphism of W onto V. An orientable surface $\mathfrak{F}$ of minimal genus properly imbedded in $\mathfrak{S}-\operatorname{Int}(fV_2)$ and spanning a longitude of $\partial(fV_2)$ may be constructed as follows: the orientable surface fS meets ∂V in w (pairwise disjoint) longitudes $\lambda_1, \cdots, \lambda_w$. Let $\mathfrak{F}_1, \cdots, \mathfrak{F}_w$ denote orientable surfaces, each of genus $g(K)$ and properly imbedded in $\mathfrak{S}-\operatorname{Int} V$, such that $\mathfrak{F}_i$ spans λ_i $(i=1, \cdots, w)$, and $\mathfrak{F}_i\cap\mathfrak{F}_j=\varnothing$ $(i\neq j)$. Evidently, $\mathfrak{F}=\mathfrak{F}_1\cup\cdots\cup\mathfrak{F}_w\cup fS$ is a surface of the type desired.

We now take a countable number of separate copies of $\mathfrak{S}-\operatorname{Int}(fV_2)$ split along $\mathfrak{F}$, and paste them together to obtain the infinite cyclic covering space $\Sigma_{\mathscr{K}}$ of $\mathfrak{S}-\operatorname{Int}(fV_2)$ [5, p. 27]. Clearly, $\Sigma_{\mathscr{K}}$ is the union of $w+1$ spaces $\mathfrak{Y}_1, \cdots, \mathfrak{Y}_w, \mathfrak{Y}_L$ such that each $\mathfrak{Y}_i$ $(i=1, \cdots, w)$ is homeomorphic to the infinite cyclic covering space of $\mathfrak{S}-\operatorname{Int} V$, $\mathfrak{Y}_i\cap\mathfrak{Y}_j=\varnothing$ $(i\neq j)$, and $\mathfrak{Y}_L$ is homeomorphic to the covering space of $W-\operatorname{Int} V_2$ obtained by properly pasting together a countable number of copies of M^3 (see hypothesis). We shall prove that $\pi_1(\Sigma_{\mathscr{K}})$ is free, whence the conclusion of Theorem 2 follows.

To begin with, $\Sigma_{\mathscr{K}}=\mathfrak{Y}_1\cup\cdots\cup\mathfrak{Y}_w\cup\mathfrak{Y}_L$, and $\mathfrak{Y}_i\cap\mathfrak{Y}_L=\mathfrak{C}_i$ $(i=1, \cdots, w)$ is a covering space of ∂V and homeomorphic to $S^1\times R$. The injections $\pi_1(\partial V)\to\pi_1(\mathfrak{S}-\operatorname{Int} V)$ and $\pi_1(\partial V)\to\pi_1(V-\operatorname{Int}(fV_2))$ are monomorphisms; the first because V is knotted [6], the second because the order w of W with respect to K_2 is greater than zero. Consequently each of the injections $\pi_1(\mathfrak{C}_i)\to\pi_1(\mathfrak{Y}_i)$ and $\pi_1(\mathfrak{C}_i)\to\pi_1(\mathfrak{Y}_L)$ $(i=1,\cdots,w)$ is a monomorphism. By an application of van Kampen's theorem [3] we may, therefore, write

$$\pi_1(\Sigma_{\mathscr{K}}) = (\cdots((\pi_1(\mathfrak{Y}_L) \underset{\pi_1(\mathfrak{C}_1)}{*} \pi_1(\mathfrak{Y}_1)) \underset{\pi_1(\mathfrak{C}_2)}{*} \pi_1(\mathfrak{Y}_2)) \underset{\pi_1(\mathfrak{C}_3)}{*} \cdots) \underset{\pi_1(\mathfrak{C}_w)}{*} \pi_1(\mathfrak{Y}_w),$$

where $\pi_1(\mathfrak{C}_i)\approx Z$ $(i=1,\cdots,w)$.

Let $\rho:\Sigma_{\mathscr{K}}\to\mathfrak{S}-\mathrm{Int}(fV_2)$ be the covering associated with the covering space $\Sigma_{\mathscr{K}}$. If the surface fS supports the group $\rho_*(\pi_1(\mathfrak{Y}_L))$ (that is, if $\rho_*(\pi_1(\mathfrak{Y}_L))=\pi_1(fS)$), then evidently $\mathfrak{F}$ supports $\rho_*(\pi_1(\Sigma_{\mathscr{K}}))$ since each $\pi_1(\mathfrak{Y}_i)$ $(i=1,\cdots,w)$ is free (hypothesis (d)). Thus, to complete the proof of Theorem 2, it suffices to prove that M^3 (see hypothesis) is homeomorphic to $S\times I$ $(I=[0, 1])$.

If M^3 is cut open along the disks $\mathscr{D}_1,\cdots,\mathscr{D}_w$ (see hypothesis (e)), we obtain a compact 3-manifold $\mathfrak{M}^3$ both homeomorphic to and a deformation retract of the space $\mathscr{M}^3$ obtained by splitting $\mathfrak{S}-\mathrm{Int}\, V_2$ along the surface S^* (hypothesis (c)). If S_1^* and S_2^* are the surfaces on $\partial\mathscr{M}^3$ closing under identification to S^*, then $\partial\mathscr{M}^3=S_1^*\cup S_2^*\cup A$. (We can assume that $S_i\subset S_i^*$ $(i=1, 2)$.) Since K_2 is a fibered knot (hypothesis (a)), $\mathscr{M}^3$ (and hence, $\mathfrak{M}^3$) is a handlebody of genus $2g(K_2)$; in fact, $\mathscr{M}^3$ is homeomorphic to $S_1^*\times I$.

Now $\partial\mathfrak{M}^3\cap\partial\mathscr{M}^3$ is a compact, connected, orientable surface of genus $2g(K_2)$ with w boundary components. We choose a canonical set Φ of $2g(K_2)$ simple closed curves on $S_1^*\cap\partial\mathfrak{M}^3\cap\partial\mathscr{M}^3$ meeting only in a common point p (the basepoint of $\pi_1(S_1^*)$). These curves (properly oriented) represent a system of free generators for $\pi_1(S_1^*)$. Now choose a system of pairwise disjoint cutting disks in $\mathscr{M}^3$, exactly one disk for each curve of our canonical set Φ of curves and of transverse intersection with it. Evidently, each map of the deformation retraction of $\mathscr{M}^3$ onto $\mathfrak{M}^3$ can be looked upon as a homeomorphism of $\mathscr{M}^3$ onto a subspace of itself leaving $\partial\mathfrak{M}^3\cap\partial\mathscr{M}^3$ pointwise fixed (one simply "pushes in" each of the w (open) disks in $\partial\mathscr{M}^3-\partial\mathfrak{M}^3$).

The deformation retraction of $\mathscr{M}^3$ onto $\mathfrak{M}^3$ leads to a complete system $\mathfrak{D}$ of cutting disks for $\mathfrak{M}^3$. It is now clear that M^3 is a handlebody of genus $w+2g(K_2)$ (cf. [8]). Hypothesis (b) implies that $S_1\cap\mathscr{C}_i$ $(i=1,\cdots,w)$ is a simple closed curve γ_i. By orienting γ_i and running an (oriented) arc δ_i from p (the common intersection of the curves in Φ) to γ_i, we obtain a collection of curves $\gamma_1\cup\delta_1,\cdots,\gamma_w\cup\delta_w$ on S_1 that, in union with Φ, represents a system of free generators for $\pi_1(S_1)$. Furthermore, each of these curves possesses transverse intersection with exactly one of the $w+2g(K_2)$ cutting disks in $\mathfrak{D}\cup\{\mathscr{D}_1,\cdots,\mathscr{D}_w\}$ of M^3. This means that the injection homomorphism $\pi_1(S_1)\to\pi_1(M^3)$ is an isomorphism (cf. [8], [13]). Similarly, the injection $\pi_1(S_2)\to\pi_1(M^3)$ is an isomorphism. It now follows from E. M. Brown's product theorem [**1**, Theorem 3.1, p. 485] that M^3 is homeomorphic to $S_1 \times I$ and, hence, to $S \times I$. This completes the proof of Theorem 2.

3. **Noninvertible fibered knots: examples.** One can readily see that T-surgery on the link of Figure 1 of [**11**] (or [**12**]) will yield an infinity of distinct noninvertible fibered knots. Figure 2 of [**11**] (Figure 3 of [**12**]) gives an example.

In order to cogently exemplify the variety of noninvertible fibered knots produced by T-surgery, we present one final example. Actually, for each positive integer n, we obtain a link L_n (Figure 1) possessing an infinite collection of prime, noninvertible fibered transmutes; one such transmute is shown in Figure 2. Note that no two links L_n and L_m are of the same type unless $n=m$ (cf. [**10**]). It is not difficult to show that the classes of transmutes (one for each L_n) are mutually disjoint. We shall omit proofs of the property claims for this example; these proofs are similar to those of the example in the above paragraph (see [**12**, §3]).

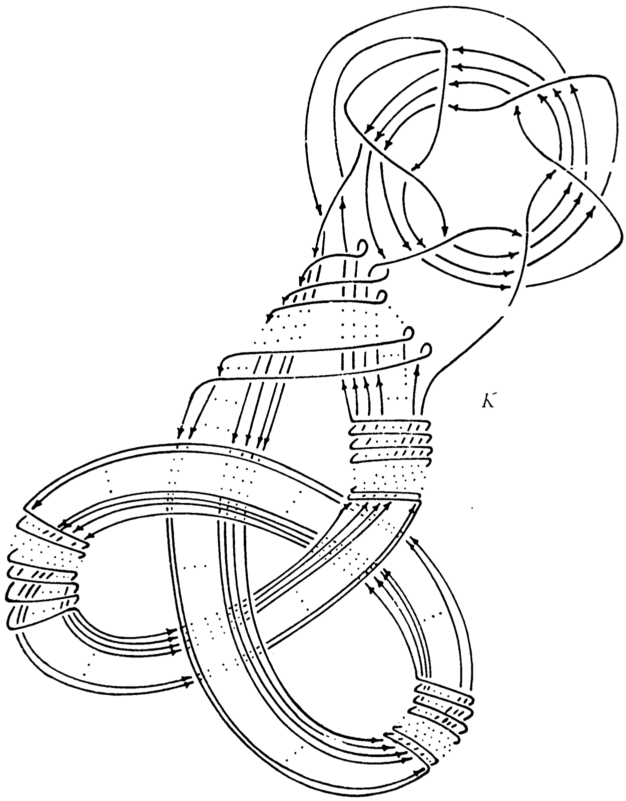

FIGURE 2

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