

$\pi_{mn-2}(S_{m-2}^n)$  CONTAINS AN ELEMENT OF ORDER  $m$

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ABSTRACT. Let  $S_m^n$  be the  $m$ th reduced product complex of  $S^n$  with  $n$  an even integer greater than 2 and  $m$  any integer greater than 2.  $S_m^n = S_{m-1}^n \cup e^{nm}$  with attaching map  $[i, \dots, i] \in \pi_{nm-1}(S_{m-1}^n)$ . Using a result of J. R. Hubbuck and a result of the author it is proven that the Whitehead product  $[i, [i, \dots, i]] \in \pi_{mn-2}(S_{m-2}^n)$  is of order  $m$ .

In [4] James defined the  $m$ th reduced product space  $X_m$  ( $m$  a positive integer or  $\infty$ ) of a complex  $X$  and proved that  $X_\infty \sim \Omega\Sigma X$ . In this note we will be concerned with the reduced product complex of spheres  $S_m^n$  for  $n \geq 4$  even and  $m \geq 3$  and will often consider  $S_m^n$  to be the  $nm$ th skeleton of  $\Omega S^{n+1}$ .

$S_m^n$  is formed from  $S_{m-1}^n$  by attaching a  $nm$ -cell (i.e.  $S_m^n = S_{m-1}^n \cup e^{nm}$ ). Call the attaching map  $[i_1, \dots, i_m] \in \pi_{nm-1}(S_{m-1}^n)$  (see [1] or [7]).

We will prove the following

THEOREM A. 1.  $\pi_{nm-1}(S_{m-1}^n) = Z \oplus T$  ( $T$  a torsion group) and  $[i, \dots, i]$  generates the infinite cyclic summand.

2. The Whitehead product  $[i, [i_1, \dots, i_{m-1}]] \in \pi_{mn-2}(S_{m-2}^n)$  is of order  $m$ . ( $i \in \pi_n(S_{m-2}^n)$  represents the inclusion  $i: S^n \rightarrow S_{m-2}^n$ .)

It should be noted that by A2,  $m[i, [i, \dots, i]] = 0$  and that this is a special case of the Jacobi identity for higher Whitehead products (Hardie [2]).

The proof of A is divided into two sections. In §1 we discuss extensions and prove all but one part of A1. §2 completes the proof.

1. **Extensions.** Let  $K = S^{n_1} \cup_{\alpha_1} e^{n_2} \cup \dots \cup_{\alpha_{r-1}} e^{n_r} \cup_{\alpha} e^n \cup_{\beta} e^m$  where  $m = n + n_1$  and  $n - 2 > n_i - 1 > n_{i-1} \geq 2$  and let  $K^i$  denote the  $i$ th skeleton of  $K$ . Let  $i \in \pi_{n_1}(K^{n-1})$  be the class of the inclusion  $i: S^{n_1} \rightarrow K^{n-1}$  and  $\sigma \in \pi_n(K^n, K^{n-1})$  be the class of the attaching map (i.e.  $\partial\sigma = \alpha$ ).

James [5, 1.4, page 262] has shown that  $\sigma_*: \pi_{m-1}(E^n, S^{n-1}) \rightarrow \pi_{m-1}(K^n, K^{n-1})$  is a monomorphism and  $\pi_{m-1}(K^n, K^{n-1})$  is isomorphic

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to the direct sum of the image of  $\sigma_*$  and an infinite cyclic subgroup generated by the relative Whitehead product  $[\sigma, i]$ .

**LEMMA 1.1.** *Let  $p:K^n \rightarrow K^n/K^{n-1} = S^n$  be the quotient map. Then there is an extension of  $p$  to  $K \rightarrow S^n$  if and only if  $i_*(\beta) \in \pi_{m-1}(K^n, K^{n-1})$  lies in the infinite cyclic summand generated by  $[\sigma, i]$ .*

**PROOF.**  $p$  extends if and only if  $p_*(\beta)$  is zero in  $\pi_{m-1}S(n)$ . Let  $p':(K^n, K^{n-1}) \rightarrow (S^n, *)$  be induced from  $p$ . The diagram

$$\begin{array}{ccc}
 \pi_{m-1}(K^n) & \xrightarrow{P_*} & \pi_{m-1}(S^{n-1}) \\
 i_* \downarrow & p'_* \nearrow & \uparrow \Sigma \partial \\
 \pi_{m-1}(K^n, K^{n-1}) & \xleftarrow{\sigma_*} & \pi_{m-1}(E^n, S^{n-1})
 \end{array}$$

is easily seen to commute. So if  $i_*(\beta) = s[\sigma, i] + \sigma_*(\rho)$  we have

$$\begin{aligned}
 p_*(\beta) &= p'_*i_*(\beta) = p'_*(s[\sigma, i] + \sigma_*(\rho)) \\
 &= sp'_*[\sigma, i] + p'_*\sigma_*(\rho) = 0 + \Sigma\partial(\rho).
 \end{aligned}$$

Since both the suspension homomorphism  $\Sigma$  and the boundary homomorphism  $\partial$  are isomorphisms  $p$  extends if and only if  $\rho$  is zero. But  $\sigma_*$  is a monomorphism and we get the required result.

Let  $\beta \in \pi_{m-1}(K^n)$ . One can (James [6]) choose generators  $x, y$  and  $z$  of dimensions  $n_1, n$  and  $m$  respectively in the integral cohomology of  $K^n \cup_{\beta} e^m$  such that if we define  $h(\beta)$ , the ‘‘Hopf invariant’’, to be that integer with  $x \cdot y = h(\beta)z, h: \pi_{m-1}(K^n) \rightarrow \mathbb{Z}$  is a homomorphism.

A simple generalization [8] of a result of James [6] yields  $i_*(\beta) = h(\beta)[\sigma, i] + \sigma_*\rho$  and by taking boundaries there exists a  $\beta$  of Hopf invariant  $m$  if and only if  $h(\beta)[\alpha, i] = \alpha_*(\tau)$ , for some  $\tau \in \pi_{m-2}(S^{n-1})$ . So that we get

**PROPOSITION 1.2.** (1)  $p:K^n \rightarrow S^n$  extends to  $K$  if and only if  $i_*(\beta) = h(\beta)[\sigma, i]$ .

(2) There exists  $\beta \in \pi_{m-1}(K^n)$  with  $h(\beta) = s$  such that  $p$  extends to  $K^n \cup_{\beta} e^m$  if and only if  $s[\alpha, i] = 0$ .

Since  $H^*(\Omega S^{n+1})$  is a divided power algebra on a generator of dimension  $n$ , we have  $h[i, \dots, i] = m$ . Further, since the map  $p: S_{m-1}^n \rightarrow S^{n(m-1)}$  extends to  $S_m^n$  we get the following special case of the Jacobi identity for higher Whitehead products [2] as a corollary of 1.2.

**COROLLARY 1.3.**  $m[i, [i_1, \dots, i_{m-1}]] = 0$ .

Thus, if we can prove part 1 of Theorem A, part 2 may be proven as follows: if  $h[i, [i, \dots, i]] = 0$  then by 1.2 there exists an element in  $\pi_{mn-1}(S_{m-1}^n)$  of Hopf invariant  $k$ . But  $h$  is a homomorphism, and any element of  $\pi_{mn-1}(S_{m-1}^n)$  is of the form  $t[i, \dots, i] + \lambda$  and  $h(t[i, \dots, i] + \lambda) = tm$  which implies  $k = tm$ .

**2. Proof of A1.** From the exact sequence

$$\dots \rightarrow \pi_i(S_\infty^n) \xrightarrow{j_*} \pi_i(S_\infty^n, S_{m-1}^n) \xrightarrow{\partial} \pi_{i-1}(S_{m-1}^n) \xrightarrow{i_*} \pi_{i-1}(S_\infty^n) \rightarrow \dots$$

we have that  $\prod_{n \geq m-1} (S_{m-1}^n) \simeq \mathbf{Z} \oplus T$  where  $T$  is a torsion group.

Since  $h$  is a homomorphism and  $h[i, \dots, i] = m$  we need only show that  $h(\gamma) \equiv 0 \pmod m$  for all  $\gamma \in \pi_{mn-1}(S_{m-1}^n)$ .

**LEMMA 2.1.** *For  $m$  an odd prime  $h(\gamma) \equiv 0 \pmod m$  ( $n \geq 4$ )*

**PROOF.** This is just 3.1 of [8].

**LEMMA 2.2 (HUBBUCK [3, 1.3]).** *Let  $Y$  be any space with  $H^*(Y; \mathbf{Z})$  free of  $p$  torsion ( $p$  a prime) and suppose that  $x \in H^n(Y; Q_p)$  ( $n > 0$ ) and  $x^p = py$  for some  $y \in H^{pn}(Y; Q_p)$  then  $y^p = pz$  for some  $z \in H^{p^2n}(Y; Q_p)$ . ( $Q_p$  is the ring of rational numbers with denominators prime to  $p$ .)*

Lemmas 2.1 and 2.2 yield

**PROPOSITION 2.3.<sup>1</sup>** *Let  $x$  generate  $H^{ni}(S_{m-1}^n \cup_\gamma e^{nm})$  and define  $k$  by  $x_1^m = kx_m$ . Then  $p^a | m!$  implies  $p^a | k$  for all primes  $p$ .*

**PROOF.** Let  $p$  be given. The proposition is proven by considering three cases.

*Case 1.* If  $m = ap$ ,  $p$  an odd prime,  $0 \leq a < p$  then  $x_1^m = (x_1^p)^a$  and the result is a direct application of 2.1.

*Case 2.* If  $m = ap^r$ ,  $r \geq 2$ ,  $0 < a < p$  then

$$x_1^{ap^r} = (x_1^{p^{r-1}})^{ap} = [(p^{r-1})! x_p^{r-1}]^{ap},$$

since the ring  $H^q(S_{m-1}^n \cup_\gamma e^{nm})$  is isomorphic to  $H^q(\Omega S^{n+1})$  for  $q < nm$ ,

$$\begin{aligned} &= (p^\beta b)^{ap} (x_p^{r-1})^{ap} \quad \text{where } \beta = 1 + p + \dots + p^{r-2} \text{ and } b \text{ is an integer} \\ &= p^{ap\beta} b^{ap} (\rho c x_p r)^a \quad \text{by 2.2 (} c \text{ an integer)} \\ &= p^{a+ap\beta} dx_m. \end{aligned}$$

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<sup>1</sup> I would like to thank Professor Gerald Porter for demonstrating the power of 2.2 in proving theorems of this type.

But the largest  $\alpha$  such that  $p^\alpha|(ap^r)!$  is given by  $\alpha = a + ap + \cdots + ap^{r-1}$  and  $a + ap\beta = a + ap(1 + \cdots + p^{r-2}) = a + ap + ap^2 + \cdots + ap^{r-1} = \alpha$  and Case 2 is proven.

Case 3. If  $m \geq 3$  is arbitrary write  $m$  in its  $p$ -adic expansion, i.e.  $m = a_0 + a_1p + \cdots + a_r p^r$  with  $0 \leq a_i < p$ . If  $a_i = 0$ ,  $i = 1, \dots, r-1$ , this is just Case 1 or Case 2 since  $a_0$  adds no powers of  $p$  to  $m!$ . So assume  $a_i \neq 0$  for some  $0 < i < r$ . Then

$$\begin{aligned} x_1^m &= x_1^{a_0} x_1^{a_1 p} \cdots x_1^{a_r p^r} = (a_0! x_{a_0}) ((a_1! p) x_{a_1 p}) \cdots ((a_r p^r)! x_{a_r p^r}) \\ &= p^{\alpha_1} \cdot p^{\alpha_2} \cdots p^{\alpha_r} x_m \end{aligned}$$

where  $\alpha_i = \sum_{j=0}^{i-1} a_j p^j$  is the largest power of  $p$  which divides  $(a_i p^i)!$ . But if  $\alpha$  is the highest power of  $p$  which divides  $m!$ ,  $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_r$  and the proposition is proven.

A1 is then proven as follows: Since (in the notation of 2.3)  $x_1^{m-1} = (m-1)! x_{m-1}$  and  $h(\gamma) x_m = x_1 x_m$  we have that  $h(\gamma) = k/(m-1)!$ ; then, by 2.3,  $m!|k$ , so that  $m|h(\gamma)$  or  $h(\gamma) \equiv 0 \pmod{m}$ .

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