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$\pi_{mn-2}(S^n_{m-2})$ CONTAINS AN ELEMENT OF ORDER m

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ABSTRACT. Let S^n_m be the mth reduced product complex of S^n with n an even integer greater than 2 and m any integer greater than 2. $S^n_m = S^n_{m-1} \cup e^{nm}$ with attaching map $[i, \cdots, i] \in \pi_{nm-1}(S^n_{m-1})$. Using a result of J. R. Hubbuck and a result of the author it is proven that the Whitehead product $[i, [i, \cdots, i]] \in \pi_{mn-2}(S^n_{m-2})$ is of order m.

In [4] James defined the mth reduced product space X_m (m a positive integer or ∞) of a complex X and proved that $X_\infty \sim \Omega\Sigma X$. In this note we will be concerned with the reduced product complex of spheres S^n_m for $n \geqq 4$ even and $m \geqq 3$ and will often consider S^n_m to be the nmth skeleton of ΩS^{n+1}.

S^n_m is formed from S^n_{m-1} by attaching a nm-call (i.e. $S^n_m = S^n_{m-1} \cup e^{nm}$). Call the attaching map $[i_1, \cdots, i_m] \in \pi_{nm-1}(S^n_{m-1})$ (see [1] or [7]).

We will prove the following

THEOREM A. 1. $\pi_{nm-1}(S^n_{m-1}) = Z \oplus T$ *(T a torsion group) and* $[i, \cdots, i]$ *generates the infinite cyclic summand.*

2. *The Whitehead product* $[i, [i_1, \cdots, i_{m-1}]] \in \pi_{mn-2}(S^n_{m-2})$ *is of order* m. *($i \in \pi_n(S^n_{m-2})$ represents the inclusion* $i: S^n \to S^n_{m-2}$.*)*

It should be noted that by A2, $m[i, [i, \cdots, i]] = 0$ and that this is a special case of the Jacobi identity for higher Whitehead products (Hardie [2]).

The proof of A is divided into two sections. In §1 we discuss extensions and prove all but one part of A1. §2 completes the proof.

1. **Extensions.** Let $K = S^{n_1} \cup_{\alpha_1} e^{n_2} \cup \cdots \cup_{\alpha_{r-1}} e^{n_r} \cup_\alpha e^n \cup_\beta e^m$ where $m = n + n_1$ and $n-2 > n_i - 1 > n_{i-1} \geqq 2$ and let K^i denote the ith skeleton of K. Let $i \in \pi_{n_1}(K^{n-1})$ be the class of the inclusion $i: S^{n_1} \to K^{n-1}$ and $\sigma \in \pi_n(K^n, K^{n-1})$ be the class of the attaching map (i.e. $\partial\sigma = \alpha$).

James [5, 1.4, page 262] has shown that $\sigma_*: \pi_{m-1}(E^n, S^{n-1}) \to \pi_{m-1}(K^n, K^{n-1})$ is a monomorphism and $\pi_{m-1}(K^n, K^{n-1})$ is isomorphic

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to the direct sum of the image of σ_* and an infinite cyclic subgroup generated by the relative Whitehead product $[\sigma, i]$.

LEMMA 1.1. *Let $p: K^n \to K^n/K^{n-1} = S^n$ be the quotient map. Then there is an extension of p to $K \to S^n$ if and only if $i_*(\beta) \in \pi_{m-1}(K^n, K^{n-1})$ lies in the infinite cyclic summand generated by $[\sigma, i]$.*

PROOF. p extends if and only if $p_*(\beta)$ is zero in $\pi_{m-1}S(^n)$. Let $p': (K^n, K^{n-1}) \to (S^n, *)$ be induced from p. The diagram

$$\begin{array}{ccc} \pi_{m-1}(K^n) & \xrightarrow{p_*} & \pi_{m-1}(S^{n-1}) \\ {\scriptstyle i_*}\downarrow & \nearrow{\scriptstyle p'_*} & \uparrow{\scriptstyle \Sigma\partial} \\ \pi_{m-1}(K^n, K^{n-1}) & \xleftarrow{\sigma_*} & \pi_{m-1}(E^n, S^{n-1}) \end{array}$$

is easily seen to commute. So if $i_*(\beta) = s[\sigma, i] + \sigma_*(\rho)$ we have

$$\begin{aligned} p_*(\beta) &= p'_* i_*(\beta) = p'_*(s[\sigma, i] + \sigma_*(\rho)) \\ &= s p'_*[\sigma, i] + p'_* \sigma_*(\rho) = 0 + \Sigma\partial(\rho). \end{aligned}$$

Since both the suspension homomorphism Σ and the boundary homomorphism ∂ are isomorphisms p extends if and only if ρ is zero. But σ_* is a monomorphism and we get the required result.

Let $\beta \in \pi_{m-1}(K^n)$. One can (James [**6**]) choose generators x, y and z of dimensions n_1, n and m respectively in the integral cohomology of $K^n \cup_\beta e^m$ such that if we define $h(\beta)$, the "Hopf invariant", to be that integer with $x \cdot y = h(\beta) z$, $h: \pi_{m-1}(K^n) \to \mathbf{Z}$ is a homomorphism.

A simple generalization [**8**] of a result of James [**6**] yields $i_*(\beta) = h(\beta)[\sigma, i] + \sigma_* \rho$ and by taking boundaries there exists a β of Hopf invariant m if and only if $h(\beta)[\alpha, i] = \alpha_*(\tau)$, for some $\tau \in \pi_{m-2}(S^{n-1})$. So that we get

PROPOSITION 1.2. (1) *$p: K^n \to S^n$ extends to K if and only if $i_*(\beta) = h(\beta)[\sigma, i]$.*

(2) *There exists $\beta \in \pi_{m-1}(K^n)$ with $h(\beta) = s$ such that p extends to $K^n \cup_\beta e^m$ if and only if $s[\alpha, i] = 0$.*

Since $H^*(\Omega S^{n+1})$ is a divided power algebra on a generator of dimension n, we have $h[i, \cdots, i] = m$. Further, since the map $p: S^n_{m-1} \to S^{n(m-1)}$ extends to S^n_m we get the following special case of the Jacobi identity for higher Whitehead products [**2**] as a corollary of 1.2.

COROLLARY 1.3. $m[i, [i_1, \cdots, i_{m-1}]] = 0$.

Thus, if we can prove part 1 of Theorem A, part 2 may be proven as follows: if $h[i, [i, \cdots, i]]=0$ then by 1.2 there exists an element in $\pi_{mn-1}(S^n_{m-1})$ of Hopf invariant k. But h is a homomorphism, and any element of $\pi_{mn-1}(S^n_{m-1})$ is of the form $t[i, \cdots, i]+\lambda$ and $h(t[i, \cdots, i]+\lambda)=tm$ which implies $k=tm$.

2. **Proof of A1.** From the exact sequence

$$\cdots \to \pi_i(S^n_\infty) \xrightarrow{j_*} \pi_i(S^n_\infty, S^n_{m-1}) \xrightarrow{\partial} n_{i-1}(S^n_{m-1}) \xrightarrow{i_*} \pi_{i-1}(S^n_\infty) \to \cdots$$

we have that $\prod_{nm-1}(S^n_{m-1}) \simeq Z \oplus T$ where T is a torsion group.

Since h is a homomorphism and $h[i, \cdots, i]=m$ we need only show that $h(\gamma)\equiv 0 \bmod m$ for all $\gamma \in \pi_{mn-1}(S^n_{m-1})$.

LEMMA 2.1. *For m an odd prime $h(\gamma)\equiv 0 \bmod m$ $(n\geqq 4)$*

PROOF. This is just 3.1 of [**8**].

LEMMA 2.2 (HUBBUCK [**3**, 1.3]). *Let Y be any space with $H^*(Y; Z)$ free of p torsion (p a prime) and suppose that $x \in H^n(Y; Q_p)$ $(n>0)$ and $x^p=py$ for some $y \in H^{pn}(Y; Q_p)$ then $y^p=pz$ for some $z \in H^{p^2n}(Y; Q_p)$. (Q_p is the ring of rational numbers with denominators prime to p.)*

Lemmas 2.1 and 2.2 yield

PROPOSITION 2.3.[1] *Let x generate $H^{ni}(S^n_{m-1} \cup_\gamma e^{nm})$ and define k by $x_1^m=kx_m$. Then $p^\alpha | m!$ implies $p^\alpha | k$ for all primes p.*

PROOF. Let p be given. The proposition is proven by considering three cases.

Case 1. If $m=ap$, p an odd prime, $0\leqq a<p$ then $x_1^m=(x_1^p)^a$ and the result is a direct application of 2.1.

Case 2. If $m=ap^r$, $r\geqq 2$, $0<a<p$ then

$$x_1^{ap^r} = (x_1^{p^{r-1}})^{ap} = [(p^{r-1})!\, x_{p^{r-1}}]^{ap},$$

since the ring $H^q(S^n_{m-1} \cup_\gamma e^{nm})$ is isomorphic to $H^q(\Omega S^{n+1})$ for $q<nm$,

$$\begin{aligned} &= (p^\beta b)^{ap}(x_{p^{r-1}})^{ap} \quad \text{where } \beta = 1 + p + \cdots + p^{r-2} \text{ and } b \text{ is an integer} \\ &= p^{ap\beta} b^{ap} (\rho c x_{p^r})^a \quad \text{by 2.2 (c an integer)} \\ &= p^{a+ap\beta}\, d x_m. \end{aligned}$$

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[1] I would like to thank Professor Gerald Porter for demonstrating the power of 2.2 in proving theorems of this type.

But the largest α such that $p^\alpha | (ap^r)!$ is given by $\alpha = a + ap + \cdots + ap^{r-1}$ and

$$a + ap\beta = a + ap(1 + \cdots + p^{r-2}) = a + ap + ap^2 + \cdots + ap^{r-1} = \alpha$$

and Case 2 is proven.

Case 3. If $m \geqq 3$ is arbitrary write m in its p-adic expansion, i.e. $m = a_0 + a_1 p + \cdots + a_r p^r$ with $0 \leqq a_i < p$. If $a_i = 0$, $i = 1, \cdots, r-1$, this is just Case 1 or Case 2 since a_0 adds no powers of p to $m!$. So assume $a_i \neq 0$ for some $0 < i < r$. Then

$$x_1^m = x_1^{a_0} x_1^{a_1 p} \cdots x_1^{a_r p^r} = (a_0! x_{a_0})((a_1! p) x_{a_1 p}) \cdots ((a_r p^r)!\, x_{a_r p^r})$$
$$= p^{\alpha_1} \cdot p^{\alpha_2} \cdots p^{\alpha_r} b x_m$$

where $\alpha_i = \sum_{j=0}^{i-1} a_i p^j$ is the largest power of p which divides $(a_i p_i)!$. But if α is the highest power of p which divides $m!$, $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ and the proposition is proven.

A1 is then proven as follows: Since (in the notation of 2.3) $x_1^{m-1} = (m-1)! x_{m-1}$ and $h(\gamma) x_m = x_1 x_m$ we have that $h(\gamma) = k/(m-1)!$; then, by 2.3, $m! | k$, so that $m | h(\gamma)$ or $h(\gamma) \equiv 0 \bmod m$.

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