\[ \pi_{m-2}(S^m_{m-2}) \text{ CONTAINS AN ELEMENT OF ORDER } m \]

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Abstract. Let \( S^n_m \) be the \( m \)th reduced product complex of \( S^n \)
with \( n \) an even integer greater than 2 and \( m \) any integer greater than 
2. \( S^n_m = S^n_{m-1} \cup e^{nm} \) with attaching map \([i, \cdots, i] \in \pi_{nm-1}(S^n_{m-1})\).
Using a result of J. R. Hubbuck and a result of the author it is
proven that the Whitehead product \([i, [i, \cdots, i]] \in \pi_{nm-4}(S^n_{m-4})\)
is of order \( m \).

In [4] James defined the \( m \)th reduced product space \( X_m \) \((m \) a positive
integer or \( \infty \)) of a complex \( X \) and proved that \( X_m \cong \Omega \Sigma X \). In this note
we will be concerned with the reduced product complex of spheres \( S^n_m \)
for \( n \geq 4 \) even and \( m \geq 3 \) and will often consider \( S^n_m \) to be the \( nm \)th skeleton
of \( \Omega S^{n+1} \).

\( S^n_m \) is formed from \( S^n_{m-1} \) by attaching a \( nm \)-call (i.e. \( S^n_m = S^n_{m-1} \cup e^{nm} \)).
Call the attaching map \([i_1, \cdots, i_m] \in \pi_{nm-1}(S^n_{m-1}) \) (see [1] or [7]).

We will prove the following.

Theorem A. 1. \( \pi_{nm-1}(S^n_{m-1}) = \mathbb{Z} \oplus T \) (\( T \) a torsion group) and \([i, \cdots, i]\)
generates the infinite cyclic summand.

2. The Whitehead product \([i, [i, \cdots, i]] \in \pi_{nm-2}(S^n_{m-2}) \) is of order \( m \). \((i \in \pi_n(S^n_{m-2}) \) represents the inclusion \( i: S^n \to S^n_{m-2} \).\)

It should be noted that by A2, \( m[i, [i, \cdots, i]] = 0 \) and that this is a
special case of the Jacobi identity for higher Whitehead products (Hardie
[2]).

The proof of A is divided into two sections. In \( \S 1 \) we discuss extensions
and prove all but one part of A1. \( \S 2 \) completes the proof.

1. Extensions. Let \( K = S^n_1 \cup e_n^1 \cup \cdots \cup e_n^{m-1} \cup e_n^m \cup e_{nm} \) where \( m = n + n_1 \) and \( n > n_1 > n_{i-1} \geq 2 \) and let \( K^i \) denote the \( i \)th skeleton of \( K \).
Let \( i \in \pi_{n_1}(K^{n-1}) \) be the class of the inclusion \( i: S^n_{m-1} \to K^{n-1} \) and
\( \sigma \in \pi_{n}(K^{n-1}) \) be the class of the attaching map (i.e. \( \partial \sigma = x \)).

James [5, 1.4, page 262] has shown that \( \sigma_*: \pi_{m-1}(E^n, S^{n-1}) \to \pi_{m-1}(K^n, K^{n-1}) \) is a monomorphism and \( \pi_{m-1}(K^n, K^{n-1}) \) is isomorphic

Received by the editors June 28, 1971 and, in revised form, October 4, 1971.

Key words and phrases. Reduced product space, extensions, Whitehead products,
Hopf invariant.
to the direct sum of the image of $\sigma_*$ and an infinite cyclic subgroup generated by the relative Whitehead product $[\sigma, i]$.

**Lemma 1.1.** Let $p : K^n \rightarrow K^n/K^{n-1} = S^n$ be the quotient map. Then there is an extension of $p$ to $K \rightarrow S^n$ if and only if $i_*(\beta) \in \pi_{m-1}(K^n, K^{n-1})$ lies in the infinite cyclic summand generated by $[\sigma, i]$.

**Proof.** $p$ extends if and only if $p_*(\beta)$ is zero in $\pi_{m-1}(S^n)$. Let $p' : (K^n, K^{n-1}) \rightarrow (S^n, *)$ be induced from $p$. The diagram

$$
\begin{array}{ccc}
\pi_{m-1}(K^n) & \xrightarrow{p_*} & \pi_{m-1}(S^n-1) \\
\downarrow i_* & & \downarrow \Sigma \partial \\
\pi_{m-1}(K^n, K^{n-1}) & \xleftarrow{\sigma_*} & \pi_{m-1}(E^n, S^{n-1})
\end{array}
$$

is easily seen to commute. So if $i_*(\beta) = s[\sigma, i] + \sigma_*(\rho)$ we have

$$p_*(\beta) = p_* i_*(\beta) = p'_*(s[\sigma, i] + \sigma_*(\rho)) = sp'_*[\sigma, i] + p'_* \sigma_*(\rho) = 0 + \Sigma \partial(\rho).$$

Since both the suspension homomorphism $\Sigma$ and the boundary homomorphism $\partial$ are isomorphisms $p$ extends if and only if $\rho$ is zero. But $\sigma_*$ is a monomorphism and we get the required result.

Let $\beta \in \pi_{m-1}(K^n)$. One can (James [6]) choose generators $x$, $y$ and $z$ of dimensions $n_1$, $n$ and $m$ respectively in the integral cohomology of $K^n \cup \rho e^m$ such that if we define $h_*(\beta)$, the “Hopf invariant”, to be that integer with $x \cdot y = h_*(\beta) z$, $h : \pi_{m-1}(K^n) \rightarrow \mathbb{Z}$ is a homomorphism.

A simple generalization [8] of a result of James [6] yields $i_*(\beta) = h_*(\beta)[\sigma, i] + \sigma_*(\rho)$ and by taking boundaries there exists a $\beta$ of Hopf invariant $m$ if and only if $h_*(\beta)[x, i] = \tau_*(\tau)$, for some $\tau \in \pi_{m-2}(S^{n-1})$. So that we get

**Proposition 1.2.**  
(1) $p : K^n \rightarrow S^n$ extends to $K$ if and only if $i_*(\beta) = h_*(\beta)[\sigma, i]$.

(2) There exists $\beta \in \pi_{m-1}(K^n)$ with $h_*(\beta) = s$ such that $p$ extends to $K^n \cup \rho e^m$ if and only if $s[x, i] = 0$.

Since $H^*(\Omega S^{n+1})$ is a divided power algebra on a generator of dimension $n$, we have $h[i, \cdots, i] = m$. Further, since the map $p : S^{n-1}_m \rightarrow S^{(m-1)}$ extends to $S^n$, we get the following special case of the Jacobi identity for higher Whitehead products [2] as a corollary of 1.2.

**Corollary 1.3.**  
$m[i, [i_1, \cdots, i_{m-1}]] = 0$. 
Thus, if we can prove part 1 of Theorem A, part 2 may be proven as follows: if \( h[i, [i, \cdots, i]] = 0 \) then by 1.2 there exists an element in \( \pi_{m-1}(S^{m-1}) \) of Hopf invariant \( k \). But \( h \) is a homomorphism, and any element of \( \pi_{m-1}(S^{m-1}) \) is of the form \( t[i, \cdots, i] + \lambda \) and \( h(t[i, \cdots, i] + \lambda) = tm \) which implies \( k = tm \).

2. Proof of A1. From the exact sequence

\[
\cdots \to \pi_i(S^n) \xrightarrow{i_*} \pi_i(S^n, S_{m-1}^n) \xrightarrow{\partial} \pi_{i-1}(S_{m-1}^n) \xrightarrow{i_*} \pi_{i-1}(S^n) \to \cdots
\]

we have that \( \prod_{i=1}^{m-1} (S_{m-1}^n) \cong \mathbb{Z} \oplus T \) where \( T \) is a torsion group.

Since \( h \) is a homomorphism and \( h[i, \cdots, i] = m \) we need only show that \( h(\gamma) \equiv 0 \mod m \) for all \( \gamma \in \pi_{m-1}(S_{m-1}^n) \).

**Lemma 2.1.** For \( m \) an odd prime \( h(\gamma) \equiv 0 \mod m \) (\( n \geq 4 \)).

**Proof.** This is just 3.1 of [8].

**Lemma 2.2** (Hubbuck [3, 1.3]). Let \( Y \) be any space with \( H^*(Y; \mathbb{Z}) \) free of \( p \) torsion (\( p \) a prime) and suppose that \( x \in H^n(Y; \mathbb{Q}_p) \) (\( n > 0 \)) and \( x^p = py \) for some \( y \in H^n(Y; \mathbb{Q}_p) \) then \( y^p = pz \) for some \( z \in H^n(Y; \mathbb{Q}_p) \). (\( \mathbb{Q}_p \) is the ring of rational numbers with denominators prime to \( p \)).

Lemmas 2.1 and 2.2 yield

**Proposition 2.3.** Let \( x \) generate \( H^{ni}(S_{m-1}^n \cup e_{nm}) \) and define \( k \) by \( x_1^m = kx_1 \). Then \( p^{|m|} \) implies \( p^{|k|} \) for all primes \( p \).

**Proof.** Let \( p \) be given. The proposition is proven by considering three cases.

Case 1. If \( m = ap \), \( p \) an odd prime, \( 0 \leq a < p \) then \( x_1^m = (x_1^a)^p \) and the result is a direct application of 2.1.

Case 2. If \( m = ap^r \), \( r \geq 2 \), \( 0 < a < p \) then

\[
x_1^{ap^r} = (x_1^{ap^{r-1}})^{ap} = [(p^{r-1})^{x_1^{ap^{r-1}}}^{ap},
\]

since the ring \( H^q(S_{m-1}^n \cup e_{nm}) \) is isomorphic to \( H^q(\Omega S^{n+1}) \) for \( q < nm \),

\[
= (p^\beta b)^{ap}(x_1^{ap^{r-1}})^{ap} \quad \text{where} \quad \beta = 1 + p + \cdots + p^{r-2} \quad \text{and} \quad b \in \mathbb{Z}
\]

\[
= p^{ap^r} b^{ap}(\rho c x_1^p)^a \quad \text{by 2.2 (c an integer)}
\]

\[
= p^{ap^r+\beta} dx_m.
\]

1 I would like to thank Professor Gerald Porter for demonstrating the power of 2.2 in proving theorems of this type.
But the largest $\alpha$ such that $p^\alpha|(ap^r)!$ is given by $\alpha = a + ap + \cdots + ap^{r-1}$ and $\alpha = a + ap(1 + \cdots + p^{r-2}) = a + ap + ap^2 + \cdots + ap^{r-1} = \alpha$

and Case 2 is proven.

Case 3. If $m \geq 3$ is arbitrary write $m$ in its $p$-adic expansion, i.e. $m = a_0 + a_1p + \cdots + a_rp^r$ with $0 \leq a_i < p$. If $a_i = 0$, $i = 1, \cdots, r - 1$, this is just Case 1 or Case 2 since $a_0$ adds no powers of $p$ to $m!$. So assume $a_i \neq 0$ for some $0 < i < r$. Then

$$x_1^m = x_1^{a_0x_1^{a_1p}} \cdots x_1^{a_rx_r} = (a_0!(x_1^p)^{(a_1!p)x_1^{a_1p}}) \cdots ((a_r!p^r)!x_1^{a_rx_r})$$

where $\alpha = \sum_{i=0}^{r-1} a_i p^i$ is the largest power of $p$ which divides $(a_i!p^i)!$. But if $\alpha$ is the highest power of $p$ which divides $m!$, $\alpha = a_1 + a_2 + \cdots + a_r$ and the proposition is proven.

A1 is then proven as follows: Since (in the notation of 2.3) $x_1^{m-1} = (m-1)!x_{m-1}$ and $h(y)x_m = x_1x_m$ we have that $h(y) = k/(m-1)!$; then, by 2.3, $m|k$, so that $m|h(y)$ or $h(y) \equiv 0 \mod m$.

REFERENCES


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