PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, July 1972

NIL ALGEBRAS SATISFYING AN IDENTITY OF DEGREE THREE

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ABSTRACT. Let A be a nonassociative algebra over a field F with a function $g:A\times A\times A\to F$ such that $(xy)z=g(x,y,z)x(yz)$ for all x, y, and z in A. Algebras satisfying this identity have been studied by Michael Rich and the author. It is shown here that a finite-dimensional nil power-associative algebra satisfying the above identity is nilpotent.

Let A be an algebra over a field F with a function $g:A\times A\times A\to F$ such that

$$(1)\qquad (xy)z = g(x,y,z)x(yz)$$

for all x, y, z in A. Semisimple algebras satisfying (1) and

$$(2)\qquad x^2x = xx^2$$

have been studied by Michael Rich and the author [2]. A power-associative algebra A is an algebra A such that for every x in A the subalgebra generated by x in A is associative. A nil power-associative algebra is an algebra in which for every x there exists an integer $n(x)$ such that $x^{n(x)}=0$. An algebra is nilpotent if there exists an integer n such that $A^n=0$. It is shown that a finite-dimensional nil power-associative algebra satisfying (1) is nilpotent. Throughout this paper we will assume that all algebras are finite dimensional and power-associative. We define the associator $(x,y,z)=(xy)z-x(yz)$. The linearization of (2) is

$$(3)\qquad (x,y,z) + (y,z,x) + (z,x,y) + (x,z,y) + (z,y,x) + (y,x,z) = 0.$$

We define $R(x)$ and $L(x)$ to be the usual endomorphisms on $AzR(x)=zx$ and $zL(x)=xz$. We can then write (1) in the form

$$(4)\qquad zR(a)R(b) = \alpha zR(ab),$$

$$(5)\qquad zL(a)R(b) = \beta zR(b)L(a)$$

Presented to the Society April 9, 1971; received by the editors June 22, 1971 and, in revised form, October 18, 1971.

AMS 1969 *subject classifications*. Primary 1720.

Key words and phrases. Power-associative, nil algebra, nilpotent algebra.

for all z, a, b in A and $\alpha=g(z, a, b)$, $\beta=g(a, z, b) \in F$. Throughout we assume that the characteristic of $F\neq 2$.

We begin with

THEOREM 1. *If A is an algebra over F satisfying* (1) *with $x^N=0$, then the ring $\mathscr{R}$ generated by the right and left multiplications by powers of x is a nilpotent ring.*

PROOF. From (1) we can write

$$zL(x^n) = \alpha zL(x)^n \tag{6}$$

for any z in A and where $\alpha \in F$ depends upon z, x and n. For any element S in $\mathscr{R}$ and z in A we can use (5) to pass all the right multiplications to the left and then use (4) and (5) to write zS as a linear combination of terms of the form $zS=\alpha zR(x^m)L(x)^r$ for some $\alpha \in F$. Now let $T \in \mathscr{R}$ and consider T^k where $k=N+2^N(N+1)$. We can write zT as a linear combination of terms of the form

$$\alpha zR(x^m)L(x)^r \tag{7}$$

where $r \geqq 2^N(N+1)$. We can choose a nonzero expression of this form such that r is minimal. From (3) we have

$$zL(x)^2 = zL(x^2) + \beta_1 zR(x^2) + \beta_2 zR(x)L(x) \tag{8}$$

where β_1, $\beta_2 \in F$. Into (7) substitute (8) for the two leftmost $L(x)$'s. By our choice of r, (7) is equal to

$$\alpha zR(x^m)L(x^2)L(x)^{r-2}. \tag{9}$$

Into (9) substitute (8) again for the two leftmost $L(x)$'s and, continuing this process, we get that (7) is equal to

$$\alpha zR(x^m)L(x^2)^{r_1}I_1(x) \tag{10}$$

when $I_1(x)$ is the identity operator or $L(x)$, depending upon whether r is even or odd. Note that (6) allows us to still utilize the choice of r and hence to conclude that r_1 is minimal. Replace x by x^2 in (8) and then use this same process to write (10) in the form $\alpha zR(x^m)L(x^4)^{r_2}I_2$ where I_2 is either the identity operator, $L(x)$, $L(x^2)$ or $L(x^2)L(x)$. In the same manner, again noting that (6) and the choice of r imply that r_{j-1} is minimal, we can write (10) in the form $\alpha zR(x^m)L(x^{n_j})^{r_j}I_j$. By the choice of r we have $n_j \geqq N$ for large enough j and this $T^k=0$.

We denote by $M(A)$ the associative algebra generated by the right and left multiplications of A. If B is any subalgebra of A then B^* is the subalgebra of $M(A)$ generated by the right and left multiplications of B, that is, every element in B^* is a linear combination of terms of the form $S_1S_2\cdots S_n$ where each S_i is a right or left multiplication by an element in B.

THEOREM 2. *Any finite-dimensional nil algebra A over F satisfying* (1) *is nilpotent.*

PROOF. As in the proof that any alternative nil algebra of finite dimension is nilpotent [**3**, p. 30], we let B be a subalgebra of A which is maximal with respect to the property that B^* is nilpotent. Since the subalgebra $\{0\}$ has the property and A is finite dimensional, such a maximal B exists. We assume that B is a proper subalgebra of A and so there exists an element x not in B such that

$$xB^* \subseteq B. \tag{11}$$

We let

$$C = B + F[x], \tag{12}$$

so that $C^* = (B+F[x])^*$. We will show that C^* is a nil algebra. Let $T \in C^*$. From (5) we can systematically pass all the right multiplications to the left and from (4) and (11) we can assume that each right multiplication is in B^*. Note that in this new expression for zT the number of factors from B^* is preserved. Hence we can assume that zT is of the form

$$\alpha z R(b_1)R(b_2)\cdots R(b_r)L(a_1)L(a_2)\cdots L(a_s)$$

where the $a_i \in B$ or Fx, $\alpha \in F$ and $z \in A$.

Now from (3) we can replace any $L(x)L(b)$ for $b \in B$ by a linear combination of terms of the form $L(b)L(x)$, $R(xb)$, $R(bx)$, $L(bx)$, $L(xb)$, $L(x)R(b)$, $L(b)R(x)$, $R(x)R(b)$, $R(b)R(x)$, $R(x)L(b)$ and $R(b)L(x)$. Utilizing (4), (5) and (11) allows us to replace $L(x)L(b)$ by a linear combination of terms of the form $L(b)L(x)$, $R(b_1')$, $L(b_2')$, $R(x)L(b_3')$, $R(b_4')L(x)$. As before, all the right multiplications can be passed to the left and each $R(x)$ is enveloped by an $R(b)$. Hence for any T in C^*, zT can be written as a linear combination of terms of the form

$$\alpha z\, R(b_1)R(b_2)\cdots R(b_r)L(b_{r+1})\cdots L(b_s)L(x)^i$$

or of the form

$$\alpha z\, R(x^t)L(b_1)L(b_2)\cdots L(b_s)L(x)^i$$

where the number of factors from B^* is constant. If $B^{*N}=0$ and $x^m=0$, we have $T^{2nm\bar{m}}=0$, where $\bar{m}$ is the index of nilpotency of the ring mentioned in Theorem 1, because we have either n factors from B^* on the left or an element from $\{L(x)\}^{\bar{m}}=0$, from Theorem 1. Hence every element of the finite-dimensional associative algebra C^* is nilpotent, and so C^* is nilpotent [**1**, p. 23]. But B was maximal with respect to the property of having B^* nilpotent, so we have a contradiction, implying that B is not a proper subalgebra of A. Thus $A=B$ and A^* is nilpotent. It follows [**3**, p. 18] that A is nilpotent.

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