

NIL ALGEBRAS SATISFYING AN IDENTITY OF DEGREE THREE

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ABSTRACT. Let A be a nonassociative algebra over a field F with a function $g: A \times A \times A \rightarrow F$ such that $(xy)z = g(x, y, z)x(yz)$ for all x, y , and z in A . Algebras satisfying this identity have been studied by Michael Rich and the author. It is shown here that a finite-dimensional nil power-associative algebra satisfying the above identity is nilpotent.

Let A be an algebra over a field F with a function $g: A \times A \times A \rightarrow F$ such that

$$(1) \quad (xy)z = g(x, y, z)x(yz)$$

for all x, y, z in A . Semisimple algebras satisfying (1) and

$$(2) \quad x^2x = xx^2$$

have been studied by Michael Rich and the author [2]. A power-associative algebra A is an algebra A such that for every x in A the subalgebra generated by x in A is associative. A nil power-associative algebra is an algebra in which for every x there exists an integer $n(x)$ such that $x^{n(x)} = 0$. An algebra is nilpotent if there exists an integer n such that $A^n = 0$. It is shown that a finite-dimensional nil power-associative algebra satisfying (1) is nilpotent. Throughout this paper we will assume that all algebras are finite dimensional and power-associative. We define the associator $(x, y, z) = (xy)z - x(yz)$. The linearization of (2) is

$$(3) \quad (x, y, z) + (y, z, x) + (z, x, y) + (x, z, y) \\
 + (z, y, x) + (y, x, z) = 0.$$

We define $R(x)$ and $L(x)$ to be the usual endomorphisms on $AzR(x) = zx$ and $zL(x) = xz$. We can then write (1) in the form

$$(4) \quad zR(a)R(b) = \alpha zR(ab),$$

$$(5) \quad zL(a)R(b) = \beta zR(b)L(a)$$

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for all z, a, b in A and $\alpha = g(z, a, b), \beta = g(a, z, b) \in F$. Throughout we assume that the characteristic of $F \neq 2$.

We begin with

THEOREM 1. *If A is an algebra over F satisfying (1) with $x^N = 0$, then the ring \mathcal{R} generated by the right and left multiplications by powers of x is a nilpotent ring.*

PROOF. From (1) we can write

$$(6) \quad zL(x^n) = \alpha zL(x)^n$$

for any z in A and where $\alpha \in F$ depends upon z, x and n . For any element S in \mathcal{R} and z in A we can use (5) to pass all the right multiplications to the left and then use (4) and (5) to write zS as a linear combination of terms of the form $zS = \alpha zR(x^m)L(x)^r$ for some $\alpha \in F$. Now let $T \in \mathcal{R}$ and consider T^k where $k = N + 2^N(N + 1)$. We can write zT as a linear combination of terms of the form

$$(7) \quad \alpha zR(x^m)L(x)^r$$

where $r \geq 2^N(N + 1)$. We can choose a nonzero expression of this form such that r is minimal. From (3) we have

$$(8) \quad zL(x)^2 = zL(x^2) + \beta_1 zR(x^2) + \beta_2 zR(x)L(x)$$

where $\beta_1, \beta_2 \in F$. Into (7) substitute (8) for the two leftmost $L(x)$'s. By our choice of r , (7) is equal to

$$(9) \quad \alpha zR(x^m)L(x^2)L(x)^{r-2}.$$

Into (9) substitute (8) again for the two leftmost $L(x)$'s and, continuing this process, we get that (7) is equal to

$$(10) \quad \alpha zR(x^m)L(x^2)^{r_1}I_1(x)$$

when $I_1(x)$ is the identity operator or $L(x)$, depending upon whether r is even or odd. Note that (6) allows us to still utilize the choice of r and hence to conclude that r_1 is minimal. Replace x by x^2 in (8) and then use this same process to write (10) in the form $\alpha zR(x^m)L(x^4)^{r_2}I_2$ where I_2 is either the identity operator, $L(x)$, $L(x^2)$ or $L(x^2)L(x)$. In the same manner, again noting that (6) and the choice of r imply that r_{j-1} is minimal, we can write (10) in the form $\alpha zR(x^m)L(x^{n_j})^{r_j}I_j$. By the choice of r we have $n_j \geq N$ for large enough j and this $T^k = 0$.

We denote by $M(A)$ the associative algebra generated by the right and left multiplications of A . If B is any subalgebra of A then B^* is the subalgebra of $M(A)$ generated by the right and left multiplications of B ,

that is, every element in B^* is a linear combination of terms of the form $S_1 S_2 \cdots S_n$ where each S_i is a right or left multiplication by an element in B .

THEOREM 2. *Any finite-dimensional nil algebra A over F satisfying (1) is nilpotent.*

PROOF. As in the proof that any alternative nil algebra of finite dimension is nilpotent [3, p. 30], we let B be a subalgebra of A which is maximal with respect to the property that B^* is nilpotent. Since the subalgebra $\{0\}$ has the property and A is finite dimensional, such a maximal B exists. We assume that B is a proper subalgebra of A and so there exists an element x not in B such that

$$(11) \quad xB^* \subseteq B.$$

We let

$$(12) \quad C = B + F[x],$$

so that $C^* = (B + F[x])^*$. We will show that C^* is a nil algebra. Let $T \in C^*$. From (5) we can systematically pass all the right multiplications to the left and from (4) and (11) we can assume that each right multiplication is in B^* . Note that in this new expression for zT the number of factors from B^* is preserved. Hence we can assume that zT is of the form

$$\alpha z R(b_1)R(b_2) \cdots R(b_r)L(a_1)L(a_2) \cdots L(a_s)$$

where the $a_i \in B$ or Fx , $\alpha \in F$ and $z \in A$.

Now from (3) we can replace any $L(x)L(b)$ for $b \in B$ by a linear combination of terms of the form $L(b)L(x)$, $R(xb)$, $R(bx)$, $L(bx)$, $L(xb)$, $L(x)R(b)$, $L(b)R(x)$, $R(x)R(b)$, $R(b)R(x)$, $R(x)L(b)$ and $R(b)L(x)$. Utilizing (4), (5) and (11) allows us to replace $L(x)L(b)$ by a linear combination of terms of the form $L(b)L(x)$, $R(b'_1)$, $L(b'_2)$, $R(x)L(b'_3)$, $R(b'_4)L(x)$. As before, all the right multiplications can be passed to the left and each $R(x)$ is enveloped by an $R(b)$. Hence for any T in C^* , zT can be written as a linear combination of terms of the form

$$\alpha z R(b_1)R(b_2) \cdots R(b_r)L(b_{r+1}) \cdots L(b_s)L(x)^i$$

or of the form

$$\alpha z R(x^t)L(b_1)L(b_2) \cdots L(b_s)L(x)^i$$

where the number of factors from B^* is constant. If $B^{*N} = 0$ and $x^m = 0$, we have $T^{2nm\bar{m}} = 0$, where \bar{m} is the index of nilpotency of the ring mentioned in Theorem 1, because we have either n factors from B^* on the left or an element from $\{L(x)\}^{\bar{m}} = 0$, from Theorem 1. Hence every element of the

finite-dimensional associative algebra C^* is nilpotent, and so C^* is nilpotent [1, p. 23]. But B was maximal with respect to the property of having B^* nilpotent, so we have a contradiction, implying that B is not a proper subalgebra of A . Thus $A=B$ and A^* is nilpotent. It follows [3, p. 18] that A is nilpotent.

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