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ON CLOSED ADDITIVE SEMIGROUPS IN E^n

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ABSTRACT. Let $C(A)$ be the closed additive semigroup generated by a set $A \subset E^n$. A simple necessary and sufficient condition on A for $C(A)$ to be a group is derived. An example which arose in the theory of random walks and stimulated these purely geometrical considerations is discussed at the end.

For an arbitrary subset A of the n-dimensional Euclidean space E^n $(n \geqq 1)$ let $S(A)$ denote the smallest additive semigroup containing A, A^- the closure of A with respect to the Euclidean topology, and $C(A) := (S(A))^-$. Obviously, $C(A)$ is the smallest closed additive semigroup containing A, i.e., the closed additive semigroup generated by A.

DEFINITION. $A \subset E^n$ is called *omnilateral*, if $A \neq \varnothing$ and for every hyperplane H through the origin with $A \not\subset H$ there are points of A in each of the two open half-spaces produced by H.

In other words, A is omnilateral if $A \neq \varnothing$ and $u \in E^n$, $x \in A$ and $ux > 0$ imply the existence of an $x^* \in A$ with $ux^* < 0$. Some immediate consequences are listed in the following lemma.

LEMMA. (1) *A is omnilateral if and only if $S(A)$ is omnilateral.* (2) *A is omnilateral if and only if A^- is omnilateral.* (3) *An additive group A is always omnilateral.* (4) *The image of an omnilateral set $A \subset E^n$ under a linear transformation from E^n to E^m is an omnilateral set in E^m.*

We say a set $A \subset E^n$ is genuinely n-dimensional if there is no hyperplane through the origin 0 containing A. A^c denotes the convex hull of A.

THEOREM 1. *A genuinely n-dimensional set A is omnilateral if and only if 0 is an inner point of A^c.*

PROOF. The if-part is trivial. Suppose, on the other hand, 0 is an exterior or a boundary point of A^c. There would be a hyperplane through

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the origin such that one of the corresponding open half-spaces contains no point of A [**3**, p. 20]. Thus A is not both omnilateral and genuinely n-dimensional.

THEOREM 2. *$C(A)$ is a group if and only if A is omnilateral.*

PROOF. The only-if-part follows from the Lemma, parts (1), (2) and (3).

To prove the if-part, one can assume that A is genuinely n-dimensional. Let $0 \neq x_0 \in C(A)$ and $\varepsilon > 0$ be given. Since 0 is an inner point of A^c there exists a $\lambda_0 > 0$ with $-\lambda_0 x_0 \in A^c$. Hence there exist a natural number k and k points $x_1, \cdots, x_k \in A$ and reals $\lambda_1 > 0, \cdots, \lambda_k > 0$ with

$$-\lambda_0 x_0 = \lambda_1 x_1 + \cdots + \lambda_k x_k.$$

By a theorem on approximations of real numbers by rational numbers (cf. for example [**4**, p. 170]) there exist rational numbers p_i/q ($i=0, \cdots, k$; p_i and q natural numbers) with

$$|p_i/q - \lambda_i| < (q \cdot q^{1/(k+1)})^{-1} \qquad (i = 0, \cdots, k),$$

where the common denominator q can be chosen arbitrarily large. If

$$\frac{1}{q^{1/(k+1)}} < \frac{\varepsilon}{(k+1)\max(|x_0|, \cdots, |x_k|)},$$

it follows that

$$|(p_i - \lambda_i q)x_i| < \varepsilon/(k+1) \qquad (i = 0, \cdots, k)$$

and thus

$$\left|\sum_{i=0}^{k} p_i x_i\right| = \left|\sum_{i=0}^{k} (p_i - \lambda_i q)x_i\right| < \varepsilon.$$

But $y(\varepsilon) := (p_0 - 1)x_0 + p_1 x_1 + \cdots + p_k x_k \in C(A)$ and thus $\lim_{\varepsilon \downarrow 0} y(\varepsilon) = -x_0 \in C(A)$. Q.E.D.

EXAMPLE. Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed n-dimensional random vectors and $S_r := X_1 + \cdots + X_r$ ($r = 1, 2, \cdots$) the associated random walk. An $x \in E^n$ is called possible if $P(|S_r - x| < \varepsilon$ for some $r) > 0$ for each $\varepsilon > 0$ ([**2**], [**1**]). The set C of all possible x is obviously the closure of the additive semigroup generated by the set

$$A := \{x \in E^n : P(|X_1 - x| < \varepsilon) > 0 \text{ for each } \varepsilon > 0\}.$$

A is omnilateral, if X_1 has expectation 0. Indeed, if A is not omnilateral there exists a $u \in E^n$ with $ux \geqq 0$ for all $x \in A$ and $uy > 0$ for some $y \in A$. Assuming that the expectation e of X_1 exists and denoting the distribution function of X_1 by F_{X_1}, we get

$$ue = u\int_{E^n}(x_1, \cdots, x_n)\, dF_{X_1}(x_1, \cdots, x_n) = \int_A ux\, dF_{X_1}(x_1, \cdots, x_n) > 0,$$

i.e. $e \neq 0$.

Thus, by Theorem 2, the set of all possible x of a random walk with expectation 0 is a group.

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