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TWO PROPERTIES OF R^N WITH A COMPACT GROUP TOPOLOGY

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ABSTRACT. We let R_c^N be a compact additive group, and we prove that if A is an R_c^N-measurable set, then one of the interiors of A and A' in the usual topology for R^N (written R_u^N) must be void. Also we show that the only functions from R^N to a Hausdorff space that are both R_u^N-continuous and R_c^N-measurable are the constant functions.

1. **Introduction.** If N is a positive integer and R_c^N a compact additive group (for the existence of which, see, for example, Halmos [**1**]), we can answer two questions (posed by Hawley [**2**] for R^1):

(i) If A and A' have nonvoid interiors in the usual topology for R^N (written as R_u^N), must A be R_c^N-nonmeasurable?

(ii) If f is R_u^N-continuous and R_c^N-continuous, must f be a constant function?

2. **Preliminaries.** If we let n be a positive integer, then the map sending x to nx, for each element x of R^N, is R^N-continuous as is also its inverse [**5**, p. 96, A] and [**4**, p. 141]. It now follows that $n\mathscr{M}=\mathscr{M}$, where $\mathscr{M}$ is the set of all R_c^N- (Borel-) measurable sets.

We have then the following lemma, the first part of which is trivial:

LEMMA 1. *All translations and positive-integer multiples of R_c^N-measurable sets are again R_c^N-measurable.*

3. **Question (i).** As R_c^N is a compact group, we can construct on it a unique normalised Haar measure μ. We define, for each positive integer n, $\mu^n(A)$ to be $\mu(nA)$ for all R_c^N-measurable sets A.

THEOREM 2. *For each positive integer n, μ^n is identical to μ.*

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PROOF. For n a positive integer, μ^n is indeed a Haar measure, since, for instance,

$$\mu^n(A + x) = \mu(nA + nx) = \mu(nA) = \mu^n(A),$$

for any R_c^N-measurable set A, and element x of R^N. However, for each n,

$$\mu^n(R^N) = \mu(nR^N) = \mu(R^N) = 1;$$

hence $\mu^n = \mu$.

The answer to question (i) is evident from the following theorem:

THEOREM 3. *An R_c^N-measurable set having nonvoid R_u^N-interior has Haar measure one for R_c^N.*

PROOF. Let A be an R_c^N-measurable set whose R_u^N-interior is nonvoid. Thus there is a nonvoid R_u^N-open ball in A, and we shift A (giving us B) so that the ball, C, is centred on $\mathbf{0}$; by Lemma 1, B is R_c^N-measurable and has the same measure as A. It is easy to show that, for positive integers n, $\bigcup_{n=1}^{\infty} nC = \lim_{n\to\infty} nC = R^N$; but $nC \subseteq nB$ for each n means that $\lim_{n\to\infty} nB = R^N$ $(n \in Z^+)$. Now by Lemma 1, nB is R_c^N-measurable for each positive integer n, and so the characteristic functions χ_{nB} will be R_c^N-measurable. We have $\lim_{n\to\infty} \chi_{nB} = \chi_{R^N}$ $(n \in Z^+)$, the integral $\int_{R^N} \chi_{R^N}\, d\mu = 1$, and $|\chi_{nB}(x)| \leqq \chi_{R^N}(x)$ for any x in R^N and positive integer n, allowing us to apply the Lebesgue Dominated Convergence Theorem to give

$$\lim_{n\to\infty} \int_{R^N} \chi_{nB}\, d\mu = \int_{R^N} \chi_{R^N}\, d\mu \qquad (n \in Z^+).$$

Therefore, by Theorem 2, $1 = \lim_{n\to\infty} \mu(nB) = \mu(B)$ $(n \in Z^+)$, and so $\mu(A) = 1$.

4. **Question (ii).** The answer to this question follows rather simply from that for its predecessor:

THEOREM 4. *The constant functions are the only maps from R^N to a Hausdorff space that are both R_u^N-continuous and R_c^N-measurable.*

PROOF. Let f be a R_u^N-continuous and R_c^N-measurable function to a Hausdorff space T, but which is not a constant function, and I_1 and I_2 be two nonvoid, disjoint subsets of T which are open relative to the image of R^N under f. The inverse maps of I_1 and I_2 under f are nonvoid, disjoint open subsets of R_u^N which are R_c^N-measurable, making the normalised Haar measure of R_c^N at least two.

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