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GROUP EXTENSIONS AND DISCRETE SUBGROUPS OF LIE GROUPS

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ABSTRACT. Let Γ be a discrete uniform subgroup of a connected simply connected solvable Lie group S. It is shown how S is essentially determined by Γ, using the point of view of group extensions.

Let Γ be a discrete uniform subgroup of a connected simply connected solvable Lie group S. The main purpose of this work is to study how S is essentially determined by Γ from the point of view of group extensions.

In [5], Mal'cev proved that Γ determines S uniquely if S is nilpotent. However, for a general solvable group, the situation seems less favorable. Nevertheless, L. Auslander (Theorem 2 in [1]) has obtained some results when Γ is the fundamental group of a nilmanifold, and in the subsequent works of Auslander and Tolimieri ([2], [8], [9]), stronger results were proved by using the notation of semisimple splittings, from which one can obtain the generalization of the result in [1] to arbitrary discrete uniform subgroups Γ. It is the purpose of this paper to revisit and prove directly this generalization by making use of the theory of group extensions as Auslander originally did in [1].

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1. Let G and H be topological groups. An extension of G by H is a pair (E, π) consisting of a topological group E which contains G as a closed normal subgroup and a continuous open homomorphism π of E onto H whose kernel is G. Two extensions (E_i, π_i) of G by H are said to be equivalent if there exists an isomorphism $\sigma: E_1 \to E_2$ of topological groups which leaves elements of G fixed and is such that $\pi_2\sigma = \pi_1$. If (E, π) is an extension of G by H, then this determines a homomorphism $\pi^0: H \to O(G)$, where $O(G)$ denotes the group $A(G)$ of all automorphisms of G modulo the inner automorphism group $I(G)$ of G. If (E_1, π_1) and (E_2, π_2) are equivalent, then $\pi_1^0 = \pi_2^0$. For any homomorphism $\varphi: H \to O(G)$, let $\mathrm{Ext}(G, H, \varphi)$ denote the set of all equivalence classes of the extensions (E, π) of G by H

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with $\pi^0=\varphi$. Let $[E, \pi]$ denote the equivalence class containing (E, π). φ is called the character of (E, π).

If G is a connected abelian Lie group and if H is a connected simply connected Lie group, then every extension (E, π) of G by H can be represented by a continuous factor set [**4**]. Hence if G is a vector group, then we may represent $\mathrm{Ext}(G, H, \varphi)$ as the cohomology group $H^2(H, G, \varphi)$ of H with coefficients in the continuous H-module G as in [**6**].

We remark also that if H is discrete, then the above representation of $\mathrm{Ext}(H, G, \varphi)$ is still valid.

2. Let Δ be a finitely generated torsion free discrete nilpotent group and let $N(\Delta)$ be the unique connected simply connected nilpotent Lie group having Δ as a uniform subgroup (see Mal′cev [**5**]).

Then the embedding $A(\Delta)\subset A(N(\Delta))$ induces a homomorphism $\xi: O(\Delta)\to O(N(\Delta))$ and we have

LEMMA. *If (D, σ) is an extension of Δ by Γ with the character φ, then there are a unique (up to equivalence) extension (D_*, σ_*) of $N(\Delta)$ by Γ with the character $\xi\circ\varphi$ and also a homomorphism $\alpha: D\to D_*$ such that the following diagram commutes*:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & D & \xrightarrow{\sigma} & \Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \| & & \\ 1 & \longrightarrow & N(\Delta) & \longrightarrow & D_* & \longrightarrow & \Gamma & \longrightarrow & 1. \end{array}$$

Moreover, if $[D, \sigma]=[D', \sigma']$, then $[D_*, \sigma_*]=[D'_*, \sigma'_*]$.

From this lemma, we can define $\Lambda: \mathrm{Ext}(\Delta, \Gamma, \varphi)\to\mathrm{Ext}(N(\Delta), \Gamma, \xi\circ\varphi)$ by $\Lambda([D, \sigma])=[D_*, \sigma_*]$.

3. Let (S, π) be an extension of a connected simply connected nilpotent Lie group N by a connected simply connected solvable Lie group H with the character $\psi: H\to O(N)$ and let L be a discrete subgroup of H. Thus in the direct product $S\times L$, define $S^*=\{(s, l)\in S\times L \mid \pi(s)=i(l)\}$, where $i: L\subset H$.

Now define $\pi^*: S^*\to L$, $\alpha: S^*\to S$ by $\pi^*(S, l)=l$, $\alpha(s, l)=S$, respectively. Then embedding N into S^* under $n\to(n, 1)$, we see that (S^*, π^*) is an extension of N by L with the character $\psi\circ i$ and that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & S^* & \xrightarrow{\pi^*} & L & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle i} & & \\ 1 & \longrightarrow & N & \longrightarrow & S & \xrightarrow{\pi} & H & \longrightarrow & 1. \end{array}$$

It is easy to see that (S^*, π^*) is unique (up to equivalence) and that if $[S_1, \pi_1]=[S_2, \pi_2]$, then $[S_1^*, \pi_1^*]=[S_2^*, \pi_2^*]$.

Hence we may define $\Omega(N, i)\colon \mathrm{Ext}(N, H, \psi) \to \mathrm{Ext}(N, L, \psi\circ i)$ by $\Omega([S, \pi]) = [S^*, \pi^*]$. Then we have

LEMMA. *If L is a discrete uniform subgroup of H, then $\Omega(N, i)$ is one-to-one.*

PROOF. Let C be the center of N and let $\psi_0\colon H \to A(C)$, $(\psi\circ i)_0\colon L \to A(C)$ be the representation induced from $\psi\colon H \to O(N)$, $\psi\circ i\colon L \to O(N)$, respectively.

Clearly $(\psi\circ i)_0 = \psi_0\circ i$. Hence $\Omega(C, i)\colon \mathrm{Ext}(C, H, \psi_0) \to \mathrm{Ext}(C, L, \psi_0\circ i)$ is defined.

We next identify $\mathrm{Ext}(C, H, \psi_0) \simeq H^2(H, C, \psi_0)$ and $\mathrm{Ext}(C, L, \psi_0\circ i) \simeq H^2(L, C, \psi_0\circ i)$. Under this identification, $\Omega(C, i) = i^*$, $i^*\colon H^2(H, C, \psi_0) \to H^2(L, C, \psi_0\circ i)$ being induced by $i\colon L \subset H$. Noting that C is a vector group, it follows from a result of van Est and Mostow (see **[3]** or **[6]**) that i^* (and hence $\Omega(C, i)$) is one-to-one.

Now since H is connected and since L is finitely generated as a fundamental group of a compact solvmanifold, we see from **[4]** that the groups $\mathrm{Ext}(C, H, \psi_0)$ and $\mathrm{Ext}(C, L, \psi_0\circ i)$ operate on the sets $\mathrm{Ext}(N, H, \psi)$ and $\mathrm{Ext}(N, L, \psi\circ i)$, respectively, and that this operation is simply transitive. It is an easy calculation to see that $\Omega(N, i)$ is equivariant with respect to these operations. Hence that $\Omega(C, i)$ is one-to-one implies that $\Omega(N, i)$ is one-to-one, proving the theorem.

4. We now prove

THEOREM (L. AUSLANDER). *Let S_1 and S_2 be connected simply connected solvable Lie groups with discrete uniform subgroups $D_1 \subset S_1$, $D_2 \subset S_2$, and assume that there is an isomorphism $\alpha\colon D_1 \to D_2$ with the following properties:*

(i) *If N_i is the nilradical of S_i $(i=1, 2)$, then α induces an isomorphism $\alpha_1\colon D_1 \cap N_1 \to D_2 \cap N_2$.*

(ii) *If $\varphi_i\colon S_i/N_i \to O(N_i)$ $(i=1, 2)$ is the homomorphism induced from the group extension $1 \to N_i \to S_i \to S_i/N_i \to 1$, then the following diagram commutes:*

$$\begin{array}{ccc} S_i/N_i & \xrightarrow{\varphi_1} & O(N_i) \\ \downarrow & & \downarrow \\ S_2/N_2 & \xrightarrow[\varphi_2]{} & O(N_2) \end{array}$$

where two vertical maps are induced by α.

Then S_1 is isomorphic with S_2.

PROOF. We identify D_1 with D_2 through α. Then under this identification, we can identify N_1 with N_2, S_1/N_1 with S_2/N_2, and φ_1 with φ_2. Thus we let $N=N_i$, $H=S_i/N_i$, $\varphi=\varphi_i$ and let $\pi_i:S_i\to H$ be the projection. Then $[S_i, \pi_i]\in \operatorname{Ext}(N, H, \varphi)$, $i=1, 2$ and we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & D & \xrightarrow{\sigma} & \Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & N & \longrightarrow & ND & \longrightarrow & ND/N & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & S_i & \xrightarrow[\pi_i]{} & H & \longrightarrow & 1 \end{array}$$

where $D=D_1=D_2$, $\Delta=N\cap D$ and $\Gamma=D/\Delta$.

Since by [7] Δ and $\Gamma=ND/N$ are uniform in N and H, respectively, it follows that $\Lambda([D, \sigma])=\Omega([S_i, \pi_i])$, $i=1, 2$, where Λ and $\Omega=\Omega(N, i)$ are defined as in §§2 and 3, respectively. But Ω is one-to-one by Theorem 1. Hence $[S_1, \pi_1]=[S_2, \pi_2]$, proving that two extensions (S_1, π_1) and (S_2, π_2) are equivalent.

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