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ON SET CORRESPONDENCES INTO UNIFORMLY CONVEX BANACH SPACES[1]

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ABSTRACT. It is proved that the values of a set-valued set function, the total variation of which is an atomless finite measure, are conditionally convex.

Let Σ be a nonempty σ-field of subsets of a set S. A (set) *correspondence*, say Γ, from Σ to a Banach space X maps, by definition, every element E of Σ to $\Gamma(E)$, a nonempty subset of X. In this paper E, F, G denote elements of Σ and x, y, z denote elements of X. If A and B are nonempty subsets of X then $A+B=\{x+y \mid x \in A \text{ and } y \in B\}$. The correspondence Γ is additive if $\Gamma(E)+\Gamma(F)=\Gamma(E\cup F)+\Gamma(E\cap F)$ for all E, F in Σ. The *total variation* (or simply variation) of Γ is the extended real valued set function, say v, on Σ defined as follows:

$$v(E) = \sup\left\{\sum |x_i| \;\middle|\; (x_i, E_i) \text{ is a finite sequence in } X \times \Sigma,\ \{E_i\} \text{ is a partition of } E \text{ and for all } i,\ x_i \in \Gamma(E_i)\right\}.$$

This is a direct generalization of the notion of total variation for Banach valued measures. Our main result is:

THEOREM 1. *Let Γ be an additive correspondence from Σ to a uniformly convex Banach space X. If the total variation of Γ is an atomless finite measure on Σ then the closure of $\Gamma(E)$ is convex for all E in Σ.* (*In particular, if $\Gamma(E)$ is closed it is convex.*)

This result is a generalization of Theorem 1.2 in [**3**]. The generalization consists of relaxing the finite dimensionality assumption on X in [**3**]. Let us recall that X is uniformly convex if for any two sequences (x_n), (y_n) in the unit ball of X, $|x_n+y_n|\to 2$ implies $|x_n-y_n|\to 0$. The spaces L^p and L^p_B are uniformly convex for $1<p<\infty$ and B a uniformly convex Banach space (see [**2**, §26.7.8]). The result holds, of course, if X is uniformly normable, i.e. there is an equivalent norm which is uniformly convex. The reason is that atomlessness and finiteness of the total variation are

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invariant under equivalence of norms. It should be mentioned that another generalization of Theorem 1.2 in [3] was given in [1]. There the assumption of closedness was dispensed with. In the end of the paper an example is given where Γ has nonconvex values because X is not uniformly convex. Our next result is:

THEOREM 2. *Under the conditions of Theorem* 1 *the closure of the range of* Γ *is convex.*

The range of the correspondence Γ is the set $\bigcup_{E\in\Sigma}\Gamma(E)$ in X. In the finite dimensional case the conclusion is stronger (see [1], [3] and [4]). Convexity of the range of Γ is a necessary tool in general equilibrium theory [4].

To simplify the proofs we start with several lemmas.

LEMMA 1. *Let* Γ *be an additive correspondence from* Σ *to a Banach space* X *and suppose that* v, *the total variation of* Γ, *is a finite measure on* Σ. *For* E *in* Σ *and* x *in* $\Gamma(E)$ *we define*

$$u(x, E) = \sup\Big\{\sum |x_i| \;\Big|\; (x_i, E_i) \text{ is a finite sequence in } X\times\Sigma,$$
$$\{E_i\} \text{ is a partition of } E,\ x_i \in \Gamma(E_i) \text{ for all } i \text{ and } \sum x_i = x\Big\}.$$

Given $\varepsilon>0$, $\delta>0$, $\bar{E}$ *in* Σ *and* $\bar{x}$ *in* $\Gamma(\bar{E})$ *we have then a finite sequence* $(\bar{x}_i, \bar{E}_i)$, *say* $i=0, 1, \cdots, k$, *in* $X\times\Sigma$ *so that* $\{\bar{E}_i\}$ *is a partition of* $\bar{E}$, $\bar{x}_i \in \Gamma(\bar{E}_i)$ *for all* i, $\sum \bar{x}_i=\bar{x}$, $u(\bar{x}_i, \bar{E}_i)-|\bar{x}_i|\leqq \varepsilon v(\bar{E}_i)$ *for* $i=1, \cdots, k$, *and* $v(\bar{E}_0)<\delta$ (*hence also* $|\bar{x}_0|<\delta$).

PROOF. First we observe that u is superadditive and bounded by v. Superadditivity here means that for all x, y, E, F such that $E\cap F=\varnothing$, $x\in\Gamma(E)$ and $y\in\Gamma(F)$ we have $u(x, E)+(u, F)\leqq u(x+y, E\cup F)$. Boundedness by v means that for all x, E with $x\in\Gamma(E)$ we have $u(x, E)\leqq v(E)$.

For any $\alpha>0$ we have, by the definition of $u(\bar{x}, \bar{E})$, a finite sequence (x_i, F_i) so that $\{F_i\}$ is a partition of $\bar{E}$, $\sum_i x_i=\bar{x}$, $u(\bar{x}, \bar{E})-\sum_i |x_i|\leqq \alpha\varepsilon v(\bar{E})$ and for all i, $x_i\in\Gamma(F_i)$. Let I be the subset of the indices $\{i\}$ such that for all i in I, $u(x_i, F_i)-|x_i|>\varepsilon v(F_i)$. Then

$$\alpha\varepsilon v(\bar{E}) \geqq u(\bar{x}, \bar{E}) - \sum_i |x_i| \geqq \sum_i (u(x_i, F_i) - |x_i|)$$
$$\geqq \sum_{i\in I} (u(x_i, F_i) - |x_i|) > \sum_{i\in I} \varepsilon v(F_i) = \varepsilon v\Big(\bigcup_{i\in I} F_i\Big).$$

(In addition to superadditivity of u the inequality $u(x_i, F_i)-|x_i|\geqq 0$ was used.) To complete the proof of the lemma we define α by the equation $\delta=\alpha v(\bar{E})$, which implies that $\delta>v(\bigcup_{i\in I} F_i)$. Also we denote $\bigcup_{i\in I} F_i$ by $\bar{E}_0$, $\sum_{i\in I} x_i$ by $\bar{x}_0$, and for all i not in I let $\bar{E}_i=F_i$ and $\bar{x}_i=x_i$. Q.E.D.

LEMMA 2. *Suppose, in addition to the assumptions of Lemma* 1, *that* v *is atomless. Then for any* $\delta>0$ *and* (x, E) *with* $x\in\Gamma(E)$ *we have* x_1, x_2, E_1, E_2 *with the following properties*: $\{E_1, E_2\}$ *is a partition of* E, $x_1+x_2=x$, $x_1\in\Gamma(E_1)$, $x_2\in\Gamma(E_2)$ *and* $|\,|x_1|-|x_2|\,|<\delta$.

PROOF. By the atomlessness assumption there is a finite partition, say $\{F_1,\cdots,F_k\}$ of E with $v(F_i)<\delta/2$ for $i=1,\cdots,k$. Let $y_i\in\Gamma(F_i)$; $i=1,\cdots,k$ and $\sum y_i=x$ (such decomposition of x exists because of the additivity of Γ). We have $|y_i|<\delta/2$ for all i. Let j be such that $|\sum_{i=1}^{j} y_i|\geqq|\sum_{i=j+1}^{k} y_i|$ but $|\sum_{i=1}^{j-1} y_i|\leqq|\sum_{i=j}^{k} y_i|$ (we use the convention that summation over an empty set gives the origin of X). Then

$$0\leqq\left|\sum_{i=j}^{k} y_i\right|-\left|\sum_{i=1}^{j-1} y_i\right|$$

$$\leqq\left(\left|\sum_{i=j+1}^{k} y_i\right|+|y_j|\right)-\left(\left|\sum_{i=1}^{j} y_i\right|-|y_j|\right)\leqq 2\,|y_j|<\delta.$$

To complete the proof we define $x_1=\sum_{i=1}^{j-1} y_i$, $x_2=\sum_{i=j}^{k} y_i$, $E_1=\bigcup_{i=1}^{j-1} F_i$ and $E_2=\bigcup_{i=j}^{k} F_i$. Q.E.D.

LEMMA 3. *Suppose, in addition to the assumptions of Lemma* 2, *that* X *is uniformly convex. Then for every* $\delta>0$ *there is* $\varepsilon>0$ *such that for any* x, y, z, E, F, G *with* $x\in\Gamma(E)$, $y\in\Gamma(F)$, $z\in\Gamma(G)$, $x=y+z$, $\{F, G\}$ *is a partition of* E, $|\,|y|-|z|\,|<\varepsilon v(E)$ *and* $u(x, E)-|x|<\varepsilon v(E)$ *we have* $|y-z|\leqq\delta v(E)$.

PROOF. If the lemma does not hold then there is $\delta>0$ and sequences (x_i), (y_i), (z_i), (E_i), (F_i), (G_i) such that for all i, $x_i\in\Gamma(E_i)$, $y_i\in\Gamma(F_i)$, $z_i\in\Gamma(G_i)$, $x_i=y_i+z_i$, $\{F_i, G_i\}$ is a partition of E_i,

$$|\,|y_i|-|z_i|\,|<v(E_i)/i,$$

$$u(x_i, E_i)-|x_i|<v(E_i)/i$$

and

$$|y_i-z_i|>\delta v(E_i).$$

We have also, for all i, $0\leqq|y_i|+|z_i|-|x_i|\leqq u(x_i, E_i)-|x_i|<v(E_i)/i$. For all i, we assume, without loss of generality, that $|y_i|\geqq|z_i|$ and we define x_i', y_i' and z_i' by the equations: $x_i'|y_i|=x_i$, $y_i'|y_i|=y_i$ and $z_i'|y_i|=z_i$. For every i,

$$\delta v(E_i)<|y_i-z_i|\leqq|y_i|+|z_i|\leqq|y_i|+|y_i|,$$

hence $|y_i|>(\frac{1}{2})\delta v(E_i)$ or $|y_i|/v(E_i)>\delta/2$. On the other hand, $|y_i|/v(E_i)\leqq 1$.

Thus, $|z_i'| \leqq |y_i'| = 1$ and

$$\begin{aligned} |x_i'| &\geqq |y_i'| + |z_i'| - v(E_i)/i\,|y_i| \\ &\geqq |y_i'| + |y_i'| - v(E_i)/i\,|y_i| - v(E_i)/i\,|y_i| \\ &= 2 - (2/i)(v(E_i)/|y_i|) \geqq 2 - (2/i)(2/\delta), \end{aligned}$$

i.e. $|x_i'| \to 2$ when $i \to \infty$. By the uniform convexity of X we have $|y_i' - z_i'| \to 0$ when $i \to \infty$. However $|y_i' - z_i'| > \delta v(E_i)/|y_i| \geqq \delta$, a contradiction. Q.E.D.

Proof of Theorem 1. In order to show the convexity of the closure of $\Gamma(\bar{E})$ it is sufficient to prove that for any $\bar{x}$, $\bar{y}$ in $\Gamma(\bar{E})$ and every $\alpha > 0$ there is $\bar{z}$ in $\Gamma(\bar{E})$ with $|(\bar{x}+\bar{y})/2 - \bar{z}| < \alpha$.

Let $X \times X$ be the Banach space with norm $|(x, y)| = (|x|^2 + |y|^2)^{1/2}$. This norm is uniformly convex because the norm of X is. For every E in Σ we define $\Gamma'(E) = \Gamma(E) \times \Gamma(E) \subset X \times X$. Let v' be the total variation of Γ'; then v' is a finite atomless measure. We have, in fact, that $v'(E) = 2^{1/2} v(E)$ for all E in Σ. We will apply Lemmas 1, 2 and 3 to Γ'.

Let $\bar{\varepsilon}$ be the $\varepsilon > 0$ corresponding to δ of Lemma 3 where $\delta > 0$ is defined by $\alpha = 4\delta v'(\bar{E})$. (We rule out the uninteresting case where $v'(\bar{E}) = 0$ because then $v(\bar{E}) = 0$ which in turn implies that $\bar{x} = \bar{y} = 0$.) Now we apply Lemma 1 to $\Gamma'(\bar{E})$ with $\varepsilon = \bar{\varepsilon}$ and $\delta = \alpha/4$. Hence we have a partition $\{E_0, E_1, \cdots, E_k\}$ of $\bar{E}$ and vectors $(x_i, x_i')_{i=0}^k$ in $X \times X$ so that $(x_i, x_i') \in \Gamma'(E_i)$ for all i, $\sum x_i = \bar{x}$, $\sum x_i' = \bar{y}$, $|(x_0, x_0')| \leqq v'(E_0) < \alpha/4$ and $u((x_i, x_i'), E_i) - |(x_i, x_i')| \leqq \bar{\varepsilon} v'(E_i)$ for $i = 1, \cdots, k$. By Lemma 2, for $i = 1, \cdots, k$, there is a partition, say $\{F_i, G_i\}$, of E_i and there are vectors (y_i, y_i') in $\Gamma'(F_i)$ and (z_i, z_i') in $\Gamma'(G_i)$ so that $(y_i, y_i') + (z_i, z_i') = (x_i, x_i')$ and $|\,|(y_i, y_i')| - |(z_i, z_i')|\,| \leqq \bar{\varepsilon} v(E_i)$. By Lemma 3, $|(y_i, y_i') - (z_i, z_i')| \leqq \delta v'(E_i) = \alpha v'(E_i)/4v'(\bar{E})$. Let $\bar{z} = x_0 + \sum_{i=1}^k y_i + \sum_{i=1}^k z_i'$. Then

$$\bar{z} \in \Gamma\left(E_0 \cup \left(\bigcup_{i=1}^k F_i\right) \cup \left(\bigcup_{i=1}^k G_i\right)\right) = \Gamma(\bar{E}).$$

To complete the proof we show that $|\bar{x} + \bar{y} - 2\bar{z}| < 2\alpha$. Indeed

$$\begin{aligned} &|\bar{x} + \bar{y} - 2\bar{z}| \\ &\quad = \left| x_0 + \sum_{i=1}^k (y_i + z_i) + x_0' + \sum_{i=1}^k (y_i' + z_i') - 2x_0 - 2\sum_{i=1}^k y_i - 2\sum_{i=1}^k z_i' \right| \\ &\quad \leqq |x_0 - x_0'| + \sum_{i=1}^k |y_i - z_i| + \sum_{i=1}^k |y_i' - z_i'| \\ &\quad \leqq |x_0| + |x_0'| + 2\delta \sum_{i=1}^k v'(E_i) \\ &\quad \leqq \alpha/4 + \alpha/4 + \alpha/2 = \alpha. \quad \text{Q.E.D.} \end{aligned}$$

Proof of Theorem 2. In order to prove the convexity of the closure of the range of Γ it is sufficient to show that for every $\bar{x}$ and $\bar{y}$ in the range of Γ and every $\alpha > 0$ there is $\bar{z}$ in the range of Γ with $|(\bar{x}+\bar{y})/2 - \bar{z}| < \alpha$. Let $\bar{x}$

be in $\Gamma(\bar{E})$ and $\bar{y}$ in $\Gamma(\bar{F})$ for some $\bar{E}$ and $\bar{F}$ in Σ. Then $x=\bar{x}_1+\bar{x}_2$ and $\bar{y}=\bar{y}_2+\bar{y}_3$ for some $\bar{x}_1$ in $\Gamma(\bar{E}\backslash\bar{F})$, $\bar{x}_2$ and $\bar{y}_2$ in $\Gamma(\bar{E}\cap\bar{F})$ and $\bar{y}_3$ in $\Gamma(\bar{F}\backslash\bar{E})$. By Theorem 1 there is $\bar{z}_2$ in $\Gamma(\bar{E}\cap\bar{F})$ with $|(\bar{x}_2+\bar{y}_2)/2-\bar{z}_2|<\alpha/3$. To complete the proof we show that there are $\bar{G}$ in Σ, $\bar{G}\subset(\bar{E}\backslash\bar{F})$ and $\bar{z}_1$ in $\Gamma(\bar{G})$ with $|\bar{x}_1/2-\bar{z}_1|<\alpha/3$.

We apply Lemma 1 to $\bar{x}_1$ in $\Gamma(\bar{E}\backslash\bar{F})$ with $\delta=\alpha/6$ and ε defined by Lemma 3 where the δ of Lemma 3 is $\alpha/(6v(\bar{E}\backslash\bar{F}))$. By Lemma 1 there is a partition $\{E_0, E_1, \cdots, E_k\}$ of $\bar{E}\backslash\bar{F}$ and vectors x_i in $\Gamma(E_i)$ with $\sum x_i=\bar{x}_1$, $|x_0|\leqq v(E_0)<\alpha/6$ and $u(x_i, E_i)-|x_i|\leqq\varepsilon v(E_i)$ for $i=1,\cdots,k$. By Lemma 2 we have, for $i=1,\cdots,k$, $\{F_i, G_i\}$ a partition of E_i, $y_i\in\Gamma(F_i)$, $z_i\in\Gamma(G_i)$, $y_i+z_i=x_i$ and $||y_i|-|z_i||<\varepsilon v(E_i)$. By Lemma 3, $|y_i-z_i|<(\alpha v(E_i))/(6v(\bar{E}\backslash\bar{F}))$. Let $\bar{z}_1=\sum_{i=1}^k z_i$. Then

$$|\bar{x}_1-2\bar{z}_1| = \left|\sum_{i=0}^k x_i - 2\sum_{i=1}^k z_i\right| = \left|x_0 + \sum_{i=1}^k (y_i+z_i) - 2\sum_{i=1}^k z_i\right|$$

$$\leqq |x_0| + \sum_{i=1}^k |y_i - z_i| \leqq \frac{\alpha}{6} + \frac{\left(\alpha \sum_{i=1}^k v(E_i)\right)}{(6v(\bar{E}\backslash\bar{F}))} \leqq \frac{\alpha}{3}. \quad \text{Q.E.D.}$$

A counterexample. Let S be the unit interval, Σ its Lebesgue measurable subsets, λ a Lebesgue measure on Σ and $X=L^1(S, \Sigma, \lambda)$. We define $\Gamma(E)=\{x_F\in X \mid F\subset E \text{ and } F\in\Sigma\}$ (x_F is the characteristic function of the set F). The variation of Γ is λ which is atomless and a finite measure and the values of Γ on Σ and its range are closed in X. But $\Gamma(E)$ is not convex for any nonnull E and the range of Γ is $\Gamma(S)$. If we consider X as $L^2(S, \Sigma, \lambda)$ then Γ is well defined but the variation of every nonnull set is infinite.

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