ON SET CORRESPONDENCES INTO UNIFORMLY
CONVEX BANACH SPACES

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Abstract. It is proved that the values of a set-valued set
function, the total variation of which is an atomless finite measure,
are conditionally convex.

Let $\Sigma$ be a nonempty $\sigma$-field of subsets of a set $S$. A (set) correspondence,
say $\Gamma$, from $\Sigma$ to a Banach space $X$ maps, by definition, every element $E$
of $\Sigma$ to $\Gamma(E)$, a nonempty subset of $X$. In this paper $E, F, G$ denote elements
of $\Sigma$ and $x, y, z$ denote elements of $X$. If $A$ and $B$ are nonempty subsets of
$X$ then $A + B = \{ x + y | x \in A$ and $y \in B \}$. The correspondence $\Gamma$ is additive
if $\Gamma(E) + \Gamma(F) = \Gamma(E \cup F) + \Gamma(E \cap F)$ for all $E, F$ in $\Sigma$. The total variation
(or simply variation) of $\Gamma$ is the extended real valued set function, say $v$,
on $\Sigma$ defined as follows:

$$v(E) = \sup \left\{ \sum |x_i| \mid (x_i, E_i) \text{ is a finite sequence in } X \times \Sigma, \right\}$$

$\{E_i\}$ is a partition of $E$ and for all $i, x_i \in \Gamma(E_i)$.

This is a direct generalization of the notion of total variation for Banach
valued measures. Our main result is:

**Theorem 1.** Let $\Gamma$ be an additive correspondence from $\Sigma$ to a uniformly
convex Banach space $X$. If the total variation of $\Gamma$ is an atomless finite
measure on $\Sigma$ then the closure of $\Gamma(E)$ is convex for all $E$ in $\Sigma$. (In particular,
if $\Gamma(E)$ is closed it is convex.)

This result is a generalization of Theorem 1.2 in [3]. The generalization
consists of relaxing the finite dimensionality assumption on $X$ in [3]. Let
us recall that $X$ is uniformly convex if for any two sequences $(x_n), (y_n)$ in
the unit ball of $X$, $|x_n + y_n| \to 2$ implies $|x_n - y_n| \to 0$. The spaces $L^p$ and $L^p_B$
are uniformly convex for $1 < p < \infty$ and $B$ a uniformly convex Banach
space (see [2, §26.7.8]). The result holds, of course, if $X$ is uniformly
normable, i.e. there is an equivalent norm which is uniformly convex.
The reason is that atomlessness and finiteness of the total variation are

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invariant under equivalence of norms. It should be mentioned that another generalization of Theorem 1.2 in [3] was given in [1]. There the assumption of closedness was dispensed with. In the end of the paper an example is given where \( \Gamma \) has nonconvex values because \( X \) is not uniformly convex. Our next result is:

**Theorem 2.** Under the conditions of Theorem 1 the closure of the range of \( \Gamma \) is convex.

The range of the correspondence \( \Gamma \) is the set \( \bigcup_{E \in \Sigma} \Gamma(E) \) in \( X \). In the finite dimensional case the conclusion is stronger (see [1], [3] and [4]). Convexity of the range of \( \Gamma \) is a necessary tool in general equilibrium theory [4].

To simplify the proofs we start with several lemmas.

**Lemma 1.** Let \( \Gamma \) be an additive correspondence from \( \Sigma \) to a Banach space \( X \) and suppose that \( v \), the total variation of \( \Gamma \), is a finite measure on \( \Sigma \). For \( E \in \Sigma \) and \( x \in \Gamma(E) \) we define

\[
u(x, E) = \sup \left\{ \sum |x_i| \mid (x_i, E_i) \text{ is a finite sequence in } \Sigma \times X, \\{E_i\} \text{ is a partition of } E, x_i \in \Gamma(E_i) \text{ for all } i \text{ and } \sum x_i = x \right\}.
\]

Given \( \varepsilon > 0 \), \( \delta > 0 \), \( E \in \Sigma \) and \( \tilde{x} \) in \( \Gamma(E) \) we have then a finite sequence \( (\tilde{x}_i, E_i) \), say \( i = 0, 1, \cdots, k \), in \( \Sigma \times X \) so that \( \{E_i\} \) is a partition of \( E \), \( \tilde{x}_i \in \Gamma(E_i) \) for all \( i \), \( \sum \tilde{x}_i = \tilde{x} \), \( u(\tilde{x}_i, E_i) - |\tilde{x}_i| \leq ev(E_i) \) for \( i = 1, \cdots, k \), and \( ev(E_0) < \delta \) (hence also \( |\tilde{x}_0| < \delta \)).

**Proof.** First we observe that \( u \) is superadditive and bounded by \( v \). Superadditivity here means that for all \( x, y, E, F \) such that \( E \cap F = \emptyset \), \( x \in \Gamma(E) \) and \( y \in \Gamma(F) \) we have \( u(x, E) + u(y, F) \leq u(x+y, E \cup F) \). Boundedness by \( v \) means that for all \( x, E \) with \( x \in \Gamma(E) \) we have \( u(x, E) \leq v(E) \).

For any \( \alpha > 0 \) we have, by the definition of \( u(\tilde{x}, E) \), a finite sequence \( (x_i, F_i) \) so that \( \{F_i\} \) is a partition of \( E \), \( \sum x_i = \tilde{x} \), \( u(\tilde{x}, E) - \sum |x_i| \leq \alpha ev(E) \) and for all \( i \), \( x_i \in \Gamma(F_i) \). Let \( I \) be the subset of the indices \( \{i\} \) such that for all \( i \) in \( I \), \( u(x_i, F_i) - |x_i| > ev(F_i) \). Then

\[
\alpha ev(E) \geq u(\tilde{x}, E) - \sum |x_i| \geq \sum (u(x_i, F_i) - |x_i|)
\]

\[
\geq \sum_{i \in I} (u(x_i, F_i) - |x_i|) > \sum_{i \in I} ev(F_i) = ev\left( \bigcup_{i \in I} F_i \right).
\]

(In addition to superadditivity of \( u \) the inequality \( u(x_i, F_i) - |x_i| \geq 0 \) was used.) To complete the proof of the lemma we define \( \delta = \alpha ev(E) \), which implies that \( \delta > v\left( \bigcup_{i \in I} F_i \right) \). Also we denote \( \bigcup_{i \in I} F_i \) by \( E_0 \), \( \sum_{i \in I} x_i \) by \( \tilde{x}_0 \), and for all \( i \) not in \( I \) let \( E_i = F_i \) and \( \tilde{x}_i = x_i \). Q.E.D.
Lemma 2. Suppose, in addition to the assumptions of Lemma 1, that $v$ is atomless. Then for any $\delta > 0$ and $(x, E)$ with $x \in \Gamma(E)$ we have $x_1, x_2, E_1, E_2$ with the following properties: $\{E_1, E_2\}$ is a partition of $E$, $x_1 + x_2 = x$, $x_1 \in \Gamma(E_1)$, $x_2 \in \Gamma(E_2)$ and $|x_1| - |x_2| < \delta$.

Proof. By the atomlessness assumption there is a finite partition, say $\{F_1, \cdots, F_k\}$ of $E$ with $v(F_i) < \delta/2$ for $i = 1, \cdots, k$. Let $y_i \in \Gamma(F_i)$; $i = 1, \cdots, k$ and $\sum y_i = x$ (such decomposition of $x$ exists because of the additivity of $\Gamma$). We have $|y_i| < \delta/2$ for all $i$. Let $j$ be such that $\sum_{i=1}^{j} y_i \geq \sum_{i=j+1}^{k} y_i$ but $\sum_{i=1}^{j} y_i \leq \sum_{i=j}^{k} y_i$ (we use the convention that summation over an empty set gives the origin of $X$). Then

$$0 \leq \left| \sum_{i=j}^{k} y_i \right| - \left| \sum_{i=1}^{j} y_i \right| \leq \left( \sum_{i=j+1}^{k} |y_i| \right) - \left( \sum_{i=1}^{j} |y_i| \right) = 2 |y_j| < \delta.$$

To complete the proof we define $x_1 = \sum_{i=1}^{j-1} y_i$, $x_2 = \sum_{i=j}^{k} y_i$, $E_1 = \bigcup_{i=1}^{j-1} F_i$ and $E_2 = \bigcup_{i=j}^{k} F_i$. Q.E.D.

Lemma 3. Suppose, in addition to the assumptions of Lemma 2, that $X$ is uniformly convex. Then for every $\delta > 0$ there is $\delta' > 0$ such that for any $x$, $y$, $z$, $E$, $F$, $G$ with $x \in \Gamma(E)$, $y \in \Gamma(F)$, $z \in \Gamma(G)$, $x = y + z$, $\{F, G\}$ is a partition of $E$, $||y| - |z|| < \delta v(E)$ and $u(x, E) - |x| < \delta v(E)$ we have $|y - z| \leq \delta v(E)$.

Proof. If the lemma does not hold then there is $\delta > 0$ and sequences $(x_i)$, $(y_i)$, $(z_i)$, $(E_i)$, $(F_i)$, $(G_i)$ such that for all $i$, $x_i \in \Gamma(E_i)$, $y_i \in \Gamma(F_i)$, $z_i \in \Gamma(G_i)$, $x_i = y_i + z_i$, $\{F_i, G_i\}$ is a partition of $E_i$,

$$||y_i| - |z_i|| < \delta v(E_i)/i,$$

$$u(x_i, E_i) - |x_i| < \delta v(E_i)/i$$

and

$$|y_i - z_i| > \delta v(E_i).$$

We have also, for all $i$, $0 \leq |y_i| + |z_i| - |x_i| \leq u(x_i, E_i) - |x_i| < \delta v(E_i)/i$. For all $i$, we assume, without loss of generality, that $|y_i| \geq |z_i|$ and we define $x'_i$, $y'_i$ and $z'_i$ by the equations: $x'_i|y'_i| = x_i$, $y'_i|y'_i| = y_i$ and $z'_i|z'_i| = z_i$. For every $i$,

$$\delta v(E_i) < |y_i - z_i| \leq |y_i| + |z_i| \leq |y_i| + |y_i|,$$

hence $|y_i| > (\frac{1}{2}) \delta v(E_i)$ or $|y_i|/v(E_i) > \delta/2$. On the other hand, $|y_i|/v(E_i) \leq 1.$
Thus, \(|z'_i| \leq |y'_i| = 1\) and
\[
|x'_i| \geq |y'_i| + |z'_i| - v(E_i)/i \quad \text{or} \quad |x'_i| \geq |y'_i| + |y'_i| - v(E_i)/i \quad \text{or} \quad |x'_i| = 2 - (2/i)(v(E_i)/|y_i|) \geq 2 - (2/i)(2/\delta),
\]
i.e. \(|x'_i| \to 2\) when \(i \to \infty\). By the uniform convexity of \(X\) we have \(|y'_i - z'_i| \to 0\)
when \(i \to \infty\). However \(|y'_i - z'_i| > \delta v(E_i)/|y_i| \geq \delta\), a contradiction. \(\text{Q.E.D.}\)

**Proof of Theorem 1.** In order to show the convexity of the closure of \(\Gamma(E)\) it is sufficient to prove that for any \(\bar{x}, \bar{y}\) in \(\Gamma(E)\) and every \(\alpha > 0\) there
is \(\bar{z}\) in \(\Gamma(E)\) with \(|(\bar{x} + \bar{y})/2 - \bar{z}| < \alpha\).

Let \(X \times X\) be the Banach space with norm \(|(x, y)| = (|x|^2 + |y|^2)^{1/2}\). This norm is unformly convex because the norm of \(X\) is. For every \(E\) in \(\Sigma\) we
define \(\Gamma'(E) = \Gamma(E) \times \Gamma(E) \subset X \times X\). Let \(v'\) be the total variation of \(\Gamma'\);
then \(v'\) is a finite atomless measure. We have, in fact, that \(v'(E) = 2^1/v(E)\) for all \(E\) in \(\Sigma\). We will apply Lemmas 1, 2 and 3 to \(\Gamma'\).

Let \(\bar{E}\) be the \(\varepsilon \geq 0\) corresponding to \(\delta\) of Lemma 3 where \(\delta > 0\) is defined
by \(\varepsilon = 4\delta v(E)\). (We rule out the uninteresting case where \(v(E') = \theta\) because
then \(v(E) = \theta\) which in turn implies that \(\bar{x} = \bar{y} = \theta\).) Now we apply Lemma 1 to \(\Gamma'(E)\) with \(\varepsilon = \bar{E}\) and \(\delta = \alpha/4\). Hence we have a partition \(\{E_0, E_1, \cdots, E_k\}\)
of \(E\) and vectors \((x_i, x'_i)\) in \(X \times X\) so that \((x_i, x'_i) \in \Gamma'(E_i)\) for all \(i\),
\[
\sum_{i=1}^k x_i = \bar{x}, \quad \sum_{i=1}^k x'_i = \bar{y}, \quad |(x_0, x'_0)| \leq v'(E_0) < \varepsilon/4 \quad \text{and} \quad u((x_i, x'_i), E_i) - |(x_i, x'_i)| \leq 2\delta v'(E_i)
\]
for \(i = 1, \cdots, k\). By Lemma 2, for \(i = 1, \cdots, k\), there is a partition,
say \(\{F_i, G_i\}\), of \(E_i\) and there are vectors \((y_i, y'_i)\) in \(\Gamma'(F_i)\) and \((z_i, z'_i)\) in \(\Gamma'(G_i)\) so that \((y_i, y'_i) + (z_i, z'_i) = (x_i, x'_i)\) and \(|(y_i, y'_i)| - |(z_i, z'_i)| \leq \delta v'(E_i)\). By Lemma 3, \(|(y_i, y'_i) - (z_i, z'_i)| \leq \delta v'(E_i) = \varepsilon v'(E_i)/4v'(E)\). Let
\[
\bar{z} = x_0 + \sum_{i=1}^k y_i + \sum_{i=1}^k z'_i.
\]
Then \(\bar{z} \in \Gamma\left(E_0 \cup \left(\bigcup_{i=1}^k F_i\right) \cup \left(\bigcup_{i=1}^k G_i\right)\right) = \Gamma(E)\).

To complete the proof we show that \(|\bar{x} + \bar{y} - 2\bar{z}| < 2\varepsilon\). Indeed
\[
|x + y - 2z| = |x_0 + \sum_{i=1}^k (y_i + z_i) + x'_0 + \sum_{i=1}^k (y'_i + z'_i) - 2x_0 - 2\sum_{i=1}^k y_i - 2\sum_{i=1}^k z'_i|
\leq |x_0 - x'_0| + \sum_{i=1}^k |y_i - z_i| + \sum_{i=1}^k |y'_i - z'_i|
\leq |x_0| + |x'_0| + 2\delta \sum_{i=1}^k v'(E_i)
\leq \alpha/4 + \alpha/4 + \alpha/2 = \alpha. \quad \text{Q.E.D.}
\]

**Proof of Theorem 2.** In order to prove the convexity of the closure of the range of \(\Gamma\) it is sufficient to show that for every \(\bar{x}\) and \(\bar{y}\) in the range of \(\Gamma\) and every \(\alpha > 0\) there is \(\bar{z}\) in the range of \(\Gamma\) with \(|(\bar{x} + \bar{y})/2 - \bar{z}| < \alpha\). Let \(\bar{x}\)
be in $\Gamma(E)$ and $\tilde{y}$ in $\Gamma(F)$ for some $E$ and $F$ in $\Sigma$. Then $x = \tilde{x}_1 + \tilde{x}_2$ and $\tilde{y} = \tilde{y}_2 + \tilde{y}_3$ for some $\tilde{x}_1$ in $\Gamma(E \setminus F)$, $\tilde{x}_2$ and $\tilde{y}_3$ in $\Gamma(E \cap F)$ and $\tilde{y}_3$ in $\Gamma(F \setminus E)$. By Theorem 1 there is $z_2$ in $\Gamma(E \cap F)$ with $|\langle \tilde{x}_2 + \tilde{y}_2 \rangle / 2 - z_2| < \alpha / 3$. To complete the proof we show that there are $G$ in $\Sigma$, $G \subset (E \setminus F)$ and $\tilde{z}_1$ in $\Gamma(G)$ with $|\tilde{x}_1 / 2 - \tilde{z}_1| < \alpha / 3$.

We apply Lemma 1 to $\tilde{x}_1$ in $\Gamma(E \setminus F)$ with $\delta = \alpha / 6$ and $\varepsilon$ defined by Lemma 3 where the $\delta$ of Lemma 3 is $\alpha / (6v(E \setminus F))$. By Lemma 1 there is a partition $\{E_0, E_1, \ldots, E_k\}$ of $E \setminus F$ and vectors $x_i$ in $\Gamma(E_i)$ with $\sum x_i = \tilde{x}_1$, $|x_0| \leq v(E_0) < \alpha / 6$ and $u(x_i, E_i) - |x_i| \leq v(E_i)$ for $i = 1, \ldots, k$. By Lemma 2 we have, for $i = 1, \ldots, k$, $\{F_i, G_i\}$ a partition of $E_i$, $y_i \in \Gamma(F_i)$, $z_i \in \Gamma(G_i)$, $y_i + z_i = x_i$ and $|y_i - z_i| < \alpha v(E_i)$. By Lemma 3, $|y_i - z_i| < \alpha v(E_i) / (6v(E \setminus F))$. Let $z_i = \sum_{i=1}^{k} z_i$. Then

$$\left| \tilde{x}_1 - 2\tilde{z}_1 \right| = \left| \sum_{i=0}^{k} x_i - 2 \sum_{i=1}^{k} z_i \right| = \left| x_0 + \sum_{i=1}^{k} (y_i + z_i) - 2 \sum_{i=1}^{k} z_i \right| \leq |x_0| + \sum_{i=1}^{k} |y_i - z_i| \leq \frac{\alpha}{6} + \frac{\alpha \sum_{i=1}^{k} v(E_i)}{6v(E \setminus F)} \leq \frac{\alpha}{3}, \quad \text{Q.E.D.}$$

A counterexample. Let $S$ be the unit interval, $\Sigma$ its Lebesgue measurable subsets, $\lambda$ a Lebesgue measure on $\Sigma$ and $X = L^1(S, \Sigma, \lambda)$. We define $\Gamma(E) = \{x_F \in X | E \subset F \text{ and } F \in \Sigma\}$ ($x_F$ is the characteristic function of the set $F$). The variation of $\Gamma$ is $\lambda$ which is atomless and a finite measure and the values of $\Gamma$ on $\Sigma$ and its range are closed in $X$. But $\Gamma(E)$ is not convex for any nonnull $E$ and the range of $\Gamma$ is $\Gamma(S)$. If we consider $X$ as $L^2(S, \Sigma, \lambda)$ then $\Gamma$ is well defined but the variation of every nonnull set is infinite.

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