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ON SEMICONTINUOUS LINEAR LATTICES

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ABSTRACT. Applying spectral theory, we proved that a linear lattice is continuous if and only if it is semicontinuous and uniformly complete. In this paper we give another proof without use of spectral theory.

Let L be a linear lattice. A sequence of elements $x_\nu \in L$ $(\nu=1, 2, \cdots)$ is called a *uniform Cauchy sequence* if there is $a \in L$ such that for any $\varepsilon>0$ we can find ν_0 for which $|x_\mu - x_\nu| \leqq \varepsilon |a|$ for μ, $\nu \geqq \nu_0$. L is said to be *uniformly complete* if every uniform Cauchy sequence is convergent. We can easily prove that every uniformly complete linear lattice is Archimedean, as done in [**1**].

A linear lattice L is said to be *semicontinuous* if every element $x \in L$ is normalable, i.e. if $\{x\}^\perp$ is a normal manifold for any $x \in L$. In [**1**] we proved the

THEOREM. *A linear lattice is continuous if and only if it is semicontinuous and uniformly complete.*

We used spectral theory to prove it. In this paper we will give another proof without use of spectral theory.

Let L be a linear lattice. If L is continuous, then L is semicontinuous by Theorem 6.15 of [**2**] and uniformly complete by Theorem 3.3 of [**2**]. Thus we will prove the converse.

We suppose that L is semicontinuous and uniformly complete. First we prove that if a sequence $0 \leqq x_\nu \uparrow_{\nu=1}^\infty$ is bounded, then there is $z \in L$ such that $[x_\nu]x \uparrow_{\nu=1}^\infty [z]x$ for $x \geqq 0$. If $x_\nu \leqq k$ for $\nu=1, 2, \cdots$, then setting

$$y_n = \sum_{\nu=1}^{n} \frac{1}{\nu}([x_\nu] - [x_{\nu-1}])k \quad \text{for } n = 1, 2, \cdots,$$

where $x_0=0$, we obtain a uniform Cauchy sequence y_n $(n=1, 2, \cdots)$. Since L is uniformly complete by assumption, there exists $z \in L$ such that $y_n \uparrow_{n=1}^\infty z$. Then $[y_n]x \uparrow_{n=1}^\infty [z]x$ for $x \geqq 0$ by Theorem 5.26 of [**2**] because

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$\{z\}^{\perp}=\{y_n:n=1,2,\cdots\}^{\perp}$. On the other hand, by Theorem 6.3 of [2] we have

$$[y_n] = \sum_{\nu=1}^{n} ([x_\nu] - [x_{\nu-1}])[k] = [x_n] \quad \text{for } n = 1, 2, \cdots .$$

Now we suppose that $0 \leqq x_\nu \downarrow_{\nu=1}^{\infty}$. For $\alpha \geqq 0$, since $(\alpha x_1 - x_\nu)^+ \uparrow_{\nu=1}^{\infty}$ and $(\alpha x_1 - x_\nu)^+ \leqq \alpha x_1$ for $\nu = 1, 2, \cdots$, there exists $z_\alpha \in L$ such that

$$[(\alpha x_1 - x_\nu)^+]x \uparrow_{\nu=1}^{\infty} [z_\alpha]x \quad \text{for } x \geqq 0,$$

as proved above. It is clear that $[z_0]=0$ and $[z_\alpha]=[x_1]$ for $\alpha>1$ because $\alpha>1$ implies $(\alpha-1)x_1 \leqq (\alpha x_1 - x_\nu)^+ \leqq \alpha x_1$. Since

$$(\alpha x_1 - x_\nu)^+ \leqq (\beta x_1 - x_\nu)^+ \quad \text{for } \alpha \leqq \beta,$$

we have $[z_\alpha] \leqq [z_\beta]$ for $0 \leqq \alpha \leqq \beta$. Since $[(\alpha x_1 - x_\nu)^+] \leqq [z_\alpha]$, we have

$$([z_\beta] - [z_\alpha])[(\alpha x_1 - x_\nu)^+] = 0 \quad \text{for } \alpha \leqq \beta.$$

Thus $([z_\beta]-[z_\alpha])(\alpha x_1 - x_\nu) = -([z_\beta]-[z_\alpha])(\alpha x_1 - x_\nu)^- \leqq 0$, i.e.

$$\alpha[(z_\beta] - [z_\alpha])x_1 \leqq ([z_\beta] - [z_\alpha])x_\nu \quad \text{for } \alpha \leqq \beta \text{ and } \nu = 1, 2, \cdots .$$

We consider a double sequence $0=\alpha_{\mu,0}<\alpha_{\mu,1}<\cdots<\alpha_{\mu,n_\mu}$ $(\mu=1, 2,\cdots)$ such that $\alpha_{\mu,\nu}-\alpha_{\mu,\nu-1}<1/\mu$ for $\nu=1,2,\cdots,n_\mu$, $\alpha_{\mu,n_\mu}>1$, and $\alpha_{\mu,\nu}$ $(\nu=1,2,\cdots,n_\mu)$ is a partial sequence of $\alpha_{\mu+1,\nu}$ $(\nu=1,2,\cdots,n_{\mu+1})$. Setting

$$y_\mu = \sum_{\nu=1}^{n_\mu} \alpha_{\mu,\nu-1}([z_{\alpha_{\mu,\nu}}] - [z_{\alpha_{\mu,\nu-1}}])x_1 \quad \text{for } \mu = 1, 2, \cdots,$$

we obtain a uniform Cauchy sequence y_μ $(\mu=1,2,\cdots)$ because

$$|y_\mu - y_\rho| \leqq (1/\mu)[z_{\alpha_{\rho,n_\rho}}]x_1 = (1/\mu)[x_1]x_1 = (1/\mu)x_1 \quad \text{for } \rho \geqq \mu.$$

Since L is uniformly complete by assumption, there exists $y \in L$ such that $\lim_{\mu\to\infty} y_\mu = y$. Since

$$\alpha_{\mu,\nu-1}([z_{\alpha_{\mu,\nu}}] - [z_{\alpha_{\mu,\nu-1}}])x_1 \leqq ([z_{\alpha_{\mu,\nu}}] - [z_{\alpha_{\mu,\nu-1}}])x_\rho,$$

we have

$$y_\mu \leqq \sum_{\nu=1}^{n_\mu} ([z_{\alpha_{\mu,\nu}}] - [z_{\alpha_{\mu,\nu-1}}])x_\rho = [z_{\alpha_{\mu,n_\mu}}]x_\rho = [x_1]x_\rho = x_\rho$$

because $[x_\rho] \leqq [x_1]$. Thus $y \leqq x_\rho$ for $\rho = 1, 2, \cdots$.

We suppose $0 \leqq z \leqq x_\nu$ for $\nu=1,2,\cdots$. Since $[(\alpha x_1 - x_\nu)^+](\alpha x_1 - x_\nu) = (\alpha x_1 - x_\nu)^+ \geqq 0$, we have

$$\alpha[z_\alpha]x_1 \geqq \alpha[(\alpha x_1 - x_\nu)^+]x_1 \geqq [(\alpha x_1 - x_\nu)^+]x_\nu \geqq [(\alpha x_1 - x_\nu)^+]z$$

for $\nu=1, 2, \cdots$, and hence $\alpha[z_\alpha]x_1 \geqq [z_\alpha]z$. Thus

$$\alpha([z_\alpha] - [z_\beta])x_1 \geqq ([z_\alpha] - [z_\beta])z \quad \text{for } \alpha \geqq \beta,$$

because $([z_\alpha]-[z_\beta])[z_\alpha]=[z_\alpha]-[z_\beta]$ by Theorem 5.24 of [2]. Since $\alpha_{\mu,\nu-1}+1/\mu \geqq \alpha_{\mu,\nu}$ $(\nu=1, 2, \cdots, n_\mu)$, we have

$$y_\mu + \frac{1}{\mu} x_1 \geqq \sum_{\nu=1}^{n_\mu} \alpha_{\mu,\nu}([z_{\alpha_{\mu,\nu}}] - [z_{\alpha_{\mu,\nu-1}}])x_1$$

$$\geqq \sum_{\nu=1}^{n_\mu} ([z_{\alpha_{\mu,\nu}}] - [z_{\alpha_{\mu,\nu-1}}])z = [x_1]z = z$$

for $\mu=1, 2, \cdots$ because $0 \leqq z \leqq x_1$ implies $[z] \leqq [x_1]$. Since L is Archimedean, $\lim_{\mu\to\infty} (1/\mu)x_1=0$, and we obtain $y \geqq z$. Therefore L is continuous by definition.

References

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