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ON TOEPLITZ OPERATORS WHICH ARE CONTRACTIONS

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ABSTRACT. We prove that a Toeplitz contraction T_ϕ is completely nonunitary if ϕ is not a constant. As an application, it is noted that for such T_ϕ, a functional calculus can be defined for all functions u in H^∞ of the unit disk.

For $1 \leqq p \leqq \infty$, we denote by L^p the usual class of Lebesgue measurable functions on the unit circle γ of the complex plane. We write H^p for the closed subspace of L^p of functions whose Fourier coefficients vanish on the negative integers. We denote by P the orthogonal projection of L^2 onto H^2 and by $B(H^2)$ the space of bounded operators on H^2. For $\phi \in L^\infty$, we consider the Toeplitz operator $T_\phi \in B(H^2)$ defined by $T_\phi f = P(\phi f)$ for $f \in H^2$. Following Sz.-Nagy and Foiaş [**2**], we say a contraction T_ϕ, $\|T_\phi\| \leqq 1$, is completely nonunitary (c.n.u.) if T_ϕ has no nontrivial reducing subspaces restricted to which T_ϕ is unitary. We will use the fact that $T_\phi^* = T_{\bar\phi}$ [**1**, p. 137].

THEOREM. *If $\phi \in L^\infty$, $\|\phi\|_\infty \leqq 1$ and ϕ is not a constant (almost everywhere), then T_ϕ is c.n.u.*

PROOF. Suppose T_ϕ is not c.n.u. We will show that ϕ is constant. Let S be a nontrivial reducing subspace for T_ϕ on which T_ϕ is unitary. We may write $S = \{f \in H^2 : \|T_\phi^n f\|_2 = \|f\|_2 = \|T_{\bar\phi}^n f\|_2 \text{ for } n = 1, 2, \cdots\}$. Now, for $f \in S$, $\|f\|_2 = \|T_\phi f\|_2 = \|P\phi f\|_2 \leqq \|\phi f\|_2 \leqq \|f\|_2$ and the resulting equality gives $\phi f \in H^2$. Similarly, $\phi^n f$ and $\bar\phi^n f \in H^2$ for $n \geqq 0$ and $f \in S$. We may apply the F. and M. Riesz theorem [**1**, p. 82] to the equality $\|\phi f\|_2 = \|f\|_2$ for a nonzero $f \in S$ to conclude that $|\phi| = 1$ almost everywhere on γ. Thus, we write $S = \{f \in H^2 : \phi^n f, \bar\phi^n f \in H^2 \text{ for } n \geqq 0\}$.

Let M_z be the operator of multiplication by the coordinate function z. Then $\phi^n f \in H^2$ implies $z\phi^n f = \phi^n z f \in H^2$ and similarly for $\bar\phi^n z f$, i.e. $M_z S \subset S$. By Beurling's theorem [**1**, p. 79], there is a function $\psi \in H^\infty$, $|\psi| = 1$ almost everywhere, such that $S = \psi H^2$. Since $1 \in H^2$, ψ is in S. Note that $\phi\psi = \psi f$ for some $f \in H^2$ (since $\phi\psi \in S = \psi H^2$). Hence for

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$n \geqq 0$, we have

$$\int_\gamma \phi z^n \, dz = \int_\gamma (\phi\psi)(\bar{\psi} z^n) \, dz = \int_\gamma (\psi f)(\bar{\psi} z^n) \, dz = \int_\gamma f z^n \, dz = 0$$

which implies that $\phi \in H^\infty$. Similarly, $\bar{\phi} \in H^\infty$ implies ϕ is a constant.

COROLLARY. *If $\|\phi\|_\infty \leqq 1$ and if ϕ is not constant, then the map $u \mapsto u(T_\phi)$ from H^∞ into $B(H^2)$ defined by*

$$u(T_\phi) = \lim_{r \to 1-0} \sum_{k=0}^{\infty} C_k r^k T_\phi^k,$$

where $u(\lambda) = \sum_{k=0}^\infty C_k \lambda^k \in H^\infty$ is a norm decreasing homomorphism of the algebra H^∞ into $B(H^2)$.

PROOF. Apply the above theorem and Theorem 2.1, Chapter III of [2].

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