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REDUCIBILITY OF ISOMETRIC IMMERSIONS

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ABSTRACT. For $i=1, 2$, suppose that the connected riemannian manifold M_i possesses a codimension p_i euclidean isometric immersion whose first normal space has dimension p_i and whose type number is at least two at each point, and let $N=\dim(M_1\times M_2)+p_1+p_2$. In this note it is proven that if f is any isometric immersion from the riemannian product $M_1\times M_2$ into euclidean N-space E^N, then there exists an orthogonal decomposition $E^N=E^{N_1}\times E^{N_2}$ together with isometric immersions $f_i: M_i\to E^{N_i}$ such that $f=f_1\times f_2$.

An isometric immersion f from a riemannian product $M_1\times M_2$ into N-dimensional euclidean space E^N is said to be *reducible* if there is an orthogonal product decomposition $E^N=E^{N_1}\times E^{N_2}$ together with isometric immersions $f_1: M_1\to E^{N_1}$ and $f_2: M_2\to E^{N_2}$ such that $f=f_1\times f_2$. It is known that if M_1 and M_2 are connected and their Riemann-Christoffel curvature tensors are nonzero almost everywhere, then every codimension two euclidean isometric immersion of $M_1\times M_2$ is reducible [**1**], [**5**]. This note is devoted to a more general reducibility theorem.

To formulate the hypothesis for our theorem, we let V denote the tangent space to a riemannian manifold M at a point m. The Riemann-Christoffel curvature tensor R at m can be regarded as an endomorphism of $V\wedge V$ which is symmetric with respect to the inner product defined by the riemannian metric. We will say that M satisfies *condition* $A(p)$ at m if there exist vectors $u, v\in V$ such that $R(u\wedge v)$ has rank at least $2p$. (Recall that $R(u\wedge v)$ has *rank* $2p$ iff p is the largest integer such that $R(u\wedge v)\wedge\cdots\wedge R(u\wedge v)$ (p times)$\neq 0$ [**3**, p. 55].)

REDUCIBILITY THEOREM. *For $i=1, 2$, let M_i be a connected riemannian manifold which satisfies condition $A(p_i)$ almost everywhere, and let $N=\dim(M_1\times M_2)+p_1+p_2$. Then any isometric immersion of the riemannian product $M_1\times M_2$ into E^N is reducible.*

Notice that if M_i possesses a codimension p_i euclidean isometric immersion f_i whose first normal space has dimension p_i and whose type number is at least two at each point (in the sense of Allendoerfer [**2**], [**4**, pp. 349–354]), then M_i satisfies condition $A(p_i)$ everywhere. In this

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case the dimension N in the Reducibility Theorem is clearly optimal, because if $g: E^N \to E^{N+1}$ is a cylindrical isometric immersion, $g \circ (f_1 \times f_2)$: $M_1 \times M_2 \to E^{N+1}$ will seldom be reducible. Reducibility is an uncommon phenomenon except in the lowest possible codimension.

M satisfies condition $A(1)$ wherever R is nonzero, i.e. at the nonflat points of M. Thus an induction based on the Reducibility Theorem and the following lemma will yield a generalization of Theorem 1 in [**5**].

LEMMA 1. *If the riemannian manifold M_i satisfies condition $A(p_i)$ at m_i for $i=1, 2$, then $M_1 \times M_2$ satisfies condition $A(p_1+p_2)$ at (m_1, m_2).*

To prove the lemma we note that the tangent space V to $M_1 \times M_2$ at (m_1, m_2) possesses an orthogonal direct sum decomposition $V = V_1 \oplus V_2$, where V_i consists of the vectors tangent to M_i. We regard $V_i \wedge V_i$ as a subspace of $V \wedge V$. If $R_i: V_i \wedge V_i \to V_i \wedge V_i$ is the curvature tensor of M_i and $\pi_i: V \wedge V \to V_i \wedge V_i$ is the orthogonal projection, then

$$R = R_1 \circ \pi_1 + R_2 \circ \pi_2 \tag{1}$$

is the curvature tensor of $M_1 \times M_2$. If condition $A(p_i)$ holds at m_i, then there exist vectors $u_i, v_i \in V_i$ such that $R_i(u_i \wedge v_i)$ has rank at least $2p_i$. By (1),

$$R((u_1 + u_2) \wedge (v_1 + v_2)) = R_1(u_1 \wedge v_1) + R_2(u_2 \wedge v_2).$$

Since $R_i(u_i \wedge v_i) \in V_i \wedge V_i$, the sum on the right has rank at least $2(p_1+p_2)$, which proves that condition $A(p_1+p_2)$ is satisfied at (m_1, m_2).

Let $n = \dim(M_1 \times M_2)$ and $p = p_1 + p_2$ so that $N = n + p$. An isometric immersion $f: M_1 \times M_2 \to E^N$ and a choice of orthonormal basis for the normal space to $M_1 \times M_2$ at (m_1, m_2) determine p second fundamental forms Φ^λ, $n+1 \leqq \lambda \leqq N$, at (m_1, m_2). The Φ^λ's are symmetric bilinear forms on the tangent space V at (m_1, m_2), and they determine symmetric endomorphisms A^λ of V by

$$\langle A^\lambda(u), v \rangle = \Phi^\lambda(u, v) \quad \text{for } u, v \in V,$$

where $\langle\ ,\ \rangle$ denotes the riemannian metric. The A^λ's in turn determine symmetric endomorphisms $A^\lambda \wedge A^\lambda$ of $V \wedge V$ which satisfy the Gauss equation

$$R = \sum_{\lambda=n+1}^{N} A^\lambda \wedge A^\lambda. \tag{2}$$

LEMMA 2. *Suppose that the second fundamental forms Φ^λ of an isometric immersion f from a connected riemannian product $M_1 \times M_2$ into E^N have the following property at every point (m_1, m_2) in $M_1 \times M_2$:*

$$\Phi^\lambda(w_1, w_2) = 0 \quad \textit{for all } w_1 \in V_1,\ w_2 \in V_2,\ n + 1 \leqq \lambda \leqq N. \tag{3}$$

Then f is reducible.

This lemma is proven in §2 of [**5**]. In order to use Lemma 2 to prove the Reducibility Theorem it suffices by continuity to show that hypothesis (3) holds at almost all points in $M_1 \times M_2$. By the hypothesis of the Reducibility Theorem and Lemma 1 it suffices to show that (3) holds at those points of $M_1 \times M_2$ at which condition $A(p)$ is satisfied.

Assume now that (m_1, m_2) is a point in $M_1 \times M_2$ at which condition $A(p)$ holds. Then we can choose vectors u, v in the tangent space V at (m_1, m_2) so that $R(u \wedge v)$ has rank at least $2p$. By equation (2),

$$R(u \wedge v) = \sum_{\lambda=n+1}^{N} A^\lambda(u) \wedge A^\lambda(v).$$

It follows that $R(u \wedge v)$ has rank exactly $2p$ and that the $2p$ vectors $A^\lambda(u)$, $A^\lambda(v)$, $n+1 \leqq \lambda \leqq N$, are linearly independent. Now let $u = u_1 + u_2$ and $v = v_1 + v_2$, where $u_i, v_i \in V_i$. Using (1) and (2) we see that

$$\sum_{\lambda=n+1}^{N} A^\lambda(u_1) \wedge A^\lambda(u) = \sum_{\lambda=n+1}^{N} A^\lambda(u_1) \wedge A^\lambda(u_2) = R(u_1 \wedge u_2) = 0,$$

and we conclude that the $A^\lambda(u_1)$'s (and hence the $A^\lambda(u_2)$'s) lie in the subspace of V generated by the $A^\lambda(u)$'s; in fact, by Cartan's lemma,

$$A^\lambda(u_1) = \sum_{\mu=n+1}^{N} c^\lambda_\mu A^\mu(u), \qquad c^\lambda_\mu = c^\mu_\lambda,$$

where the c^λ_μ's are real numbers. Similarly we can show that the $A^\lambda(v_1)$'s and the $A^\lambda(v_2)$'s lie in the span of the $A^\lambda(v)$'s. After a possible change of orthonormal basis for the normal space we can arrange that

$$A^\lambda(u_1) = c_\lambda A^\lambda(u), \qquad A^\lambda(u_2) = (1 - c_\lambda) A^\lambda(u), \tag{4}$$

where $c_\lambda = 1$ for $n+1 \leqq \lambda \leqq q$, $c_\lambda \neq 0, 1$ for $q+1 \leqq \lambda \leqq r$, and $c_\lambda = 0$ for $r+1 \leqq \lambda \leqq N$. Equations (1), (2), and (4) now imply that

$$\sum_{\lambda=n+1}^{r} c_\lambda A^\lambda(u) \wedge A^\lambda(v_2) = \sum_{\lambda=n+1}^{N} A^\lambda(u_1) \wedge A^\lambda(v_2) = R(u_1 \wedge v_2) = 0,$$

$$\sum_{\lambda=q+1}^{N} A^\lambda(v_1) \wedge (1 - c_\lambda) A^\lambda(u) = 0.$$

Therefore $A^{n+1}(v_2), \cdots, A^r(v_2)$ and $A^{q+1}(v_1), \cdots, A^N(v_1)$ are in the span of the $A^\lambda(u)$'s. But they are also in the span of the $A^\lambda(v)$'s, and since the $A^\lambda(u)$'s and the $A^\lambda(v)$'s are linearly independent we must have

$$A^\lambda(v_2) = 0 \text{ for } n+1 \leqq \lambda \leqq r, \qquad A^\lambda(v_1) = 0 \text{ for } q+1 \leqq \lambda \leqq N. \tag{5}$$

In particular the vectors $A^\lambda(v)$, $q+1 \leqq \lambda \leqq r$, must vanish, and since these vectors are linearly independent, $q=r$.

We adopt the following index conventions: $n+1 \leqq \alpha \leqq q$, $q+1 \leqq \rho \leqq N$. Then $c_\alpha=1$, $c_\rho=0$, and it follows from (4) and (5) that

$$A^\alpha(u_1) = A^\alpha(u), \qquad A^\alpha(v_1) = A^\alpha(v), \qquad A^\rho(u_1) = A^\rho(v_1) = 0.$$

Hence if $w_2 \in V_2$,

$$\sum_\alpha A^\alpha(u) \wedge A^\alpha(w_2) = \sum_{\lambda=n+1}^{N} A^\lambda(u_1) \wedge A^\lambda(w_2) = R(u_1 \wedge w_2) = 0,$$

$$\sum_\alpha A^\alpha(v) \wedge A^\alpha(w_2) = 0.$$

Since the $2(q-n)$ vectors $A^\alpha(u)$, $A^\alpha(v)$ are linearly independent, we can conclude that $A^\alpha(w_2)=0$. Similarly we can show that $A^\rho(w_1)=0$ when $w_1 \in V_1$. Therefore

$$\Phi^\alpha(w_1, w_2) = \langle w_1, A^\alpha(w_2)\rangle = 0, \qquad \Phi^\rho(w_1, w_2) = \langle A^\rho(w_1), w_2\rangle = 0,$$

for $w_1 \in V_1$, $w_2 \in V_2$. This establishes (3) at (m_1, m_2) and finishes the proof of the Reducibility Theorem.

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