

## REDUCIBILITY OF ISOMETRIC IMMERSIONS

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**ABSTRACT.** For  $i=1, 2$ , suppose that the connected riemannian manifold  $M_i$  possesses a codimension  $p_i$  euclidean isometric immersion whose first normal space has dimension  $p_i$  and whose type number is at least two at each point, and let  $N=\dim(M_1 \times M_2) + p_1 + p_2$ . In this note it is proven that if  $f$  is any isometric immersion from the riemannian product  $M_1 \times M_2$  into euclidean  $N$ -space  $E^N$ , then there exists an orthogonal decomposition  $E^N = E^{N_1} \times E^{N_2}$  together with isometric immersions  $f_i: M_i \rightarrow E^{N_i}$  such that  $f=f_1 \times f_2$ .

An isometric immersion  $f$  from a riemannian product  $M_1 \times M_2$  into  $N$ -dimensional euclidean space  $E^N$  is said to be *reducible* if there is an orthogonal product decomposition  $E^N = E^{N_1} \times E^{N_2}$  together with isometric immersions  $f_1: M_1 \rightarrow E^{N_1}$  and  $f_2: M_2 \rightarrow E^{N_2}$  such that  $f=f_1 \times f_2$ . It is known that if  $M_1$  and  $M_2$  are connected and their Riemann-Christoffel curvature tensors are nonzero almost everywhere, then every codimension two euclidean isometric immersion of  $M_1 \times M_2$  is reducible [1], [5]. This note is devoted to a more general reducibility theorem.

To formulate the hypothesis for our theorem, we let  $V$  denote the tangent space to a riemannian manifold  $M$  at a point  $m$ . The Riemann-Christoffel curvature tensor  $R$  at  $m$  can be regarded as an endomorphism of  $V \wedge V$  which is symmetric with respect to the inner product defined by the riemannian metric. We will say that  $M$  satisfies *condition A(p)* at  $m$  if there exist vectors  $u, v \in V$  such that  $R(u \wedge v)$  has rank at least  $2p$ . (Recall that  $R(u \wedge v)$  has rank  $2p$  iff  $p$  is the largest integer such that  $R(u \wedge v) \wedge \cdots \wedge R(u \wedge v)$  ( $p$  times)  $\neq 0$  [3, p. 55].)

**REDUCIBILITY THEOREM.** For  $i=1, 2$ , let  $M_i$  be a connected riemannian manifold which satisfies condition  $A(p_i)$  almost everywhere, and let  $N = \dim(M_1 \times M_2) + p_1 + p_2$ . Then any isometric immersion of the riemannian product  $M_1 \times M_2$  into  $E^N$  is reducible.

Notice that if  $M_i$  possesses a codimension  $p_i$  euclidean isometric immersion  $f_i$  whose first normal space has dimension  $p_i$  and whose type number is at least two at each point (in the sense of Allendoerfer [2], [4, pp. 349–354]), then  $M_i$  satisfies condition  $A(p_i)$  everywhere. In this

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case the dimension  $N$  in the Reducibility Theorem is clearly optimal, because if  $g: E^N \rightarrow E^{N+1}$  is a cylindrical isometric immersion,  $g \circ (f_1 \times f_2): M_1 \times M_2 \rightarrow E^{N+1}$  will seldom be reducible. Reducibility is an uncommon phenomenon except in the lowest possible codimension.

$M$  satisfies condition  $A(1)$  wherever  $R$  is nonzero, i.e. at the nonflat points of  $M$ . Thus an induction based on the Reducibility Theorem and the following lemma will yield a generalization of Theorem 1 in [5].

LEMMA 1. *If the riemannian manifold  $M_i$  satisfies condition  $A(p_i)$  at  $m_i$  for  $i=1, 2$ , then  $M_1 \times M_2$  satisfies condition  $A(p_1+p_2)$  at  $(m_1, m_2)$ .*

To prove the lemma we note that the tangent space  $V$  to  $M_1 \times M_2$  at  $(m_1, m_2)$  possesses an orthogonal direct sum decomposition  $V=V_1 \oplus V_2$ , where  $V_i$  consists of the vectors tangent to  $M_i$ . We regard  $V_i \wedge V_i$  as a subspace of  $V \wedge V$ . If  $R_i: V_i \wedge V_i \rightarrow V_i \wedge V_i$  is the curvature tensor of  $M_i$  and  $\pi_i: V \wedge V \rightarrow V_i \wedge V_i$  is the orthogonal projection, then

$$(1) \quad R = R_1 \circ \pi_1 + R_2 \circ \pi_2$$

is the curvature tensor of  $M_1 \times M_2$ . If condition  $A(p_i)$  holds at  $m_i$ , then there exist vectors  $u_i, v_i \in V_i$  such that  $R_i(u_i \wedge v_i)$  has rank at least  $2p_i$ . By (1),

$$R((u_1 + u_2) \wedge (v_1 + v_2)) = R_1(u_1 \wedge v_1) + R_2(u_2 \wedge v_2).$$

Since  $R_i(u_i \wedge v_i) \in V_i \wedge V_i$ , the sum on the right has rank at least  $2(p_1+p_2)$ , which proves that condition  $A(p_1+p_2)$  is satisfied at  $(m_1, m_2)$ .

Let  $n=\dim(M_1 \times M_2)$  and  $p=p_1+p_2$  so that  $N=n+p$ . An isometric immersion  $f: M_1 \times M_2 \rightarrow E^N$  and a choice of orthonormal basis for the normal space to  $M_1 \times M_2$  at  $(m_1, m_2)$  determine  $p$  second fundamental forms  $\Phi^\lambda$ ,  $n+1 \leq \lambda \leq N$ , at  $(m_1, m_2)$ . The  $\Phi^\lambda$ 's are symmetric bilinear forms on the tangent space  $V$  at  $(m_1, m_2)$ , and they determine symmetric endomorphisms  $A^\lambda$  of  $V$  by

$$\langle A^\lambda(u), v \rangle = \Phi^\lambda(u, v) \quad \text{for } u, v \in V,$$

where  $\langle \ , \ \rangle$  denotes the riemannian metric. The  $A^\lambda$ 's in turn determine symmetric endomorphisms  $A^\lambda \wedge A^\lambda$  of  $V \wedge V$  which satisfy the Gauss equation

$$(2) \quad R = \sum_{\lambda=n+1}^N A^\lambda \wedge A^\lambda.$$

LEMMA 2. *Suppose that the second fundamental forms  $\Phi^\lambda$  of an isometric immersion  $f$  from a connected riemannian product  $M_1 \times M_2$  into  $E^N$  have the following property at every point  $(m_1, m_2)$  in  $M_1 \times M_2$ :*

$$(3) \quad \Phi^\lambda(w_1, w_2) = 0 \quad \text{for all } w_1 \in V_1, w_2 \in V_2, n+1 \leq \lambda \leq N.$$

*Then  $f$  is reducible.*

This lemma is proven in §2 of [5]. In order to use Lemma 2 to prove the Reducibility Theorem it suffices by continuity to show that hypothesis (3) holds at almost all points in  $M_1 \times M_2$ . By the hypothesis of the Reducibility Theorem and Lemma 1 it suffices to show that (3) holds at those points of  $M_1 \times M_2$  at which condition  $A(p)$  is satisfied.

Assume now that  $(m_1, m_2)$  is a point in  $M_1 \times M_2$  at which condition  $A(p)$  holds. Then we can choose vectors  $u, v$  in the tangent space  $V$  at  $(m_1, m_2)$  so that  $R(u \wedge v)$  has rank at least  $2p$ . By equation (2),

$$R(u \wedge v) = \sum_{\lambda=n+1}^N A^\lambda(u) \wedge A^\lambda(v).$$

It follows that  $R(u \wedge v)$  has rank exactly  $2p$  and that the  $2p$  vectors  $A^\lambda(u), A^\lambda(v)$ ,  $n+1 \leq \lambda \leq N$ , are linearly independent. Now let  $u = u_1 + u_2$  and  $v = v_1 + v_2$ , where  $u_i, v_i \in V_i$ . Using (1) and (2) we see that

$$\sum_{\lambda=n+1}^N A^\lambda(u_1) \wedge A^\lambda(u) = \sum_{\lambda=n+1}^N A^\lambda(u_1) \wedge A^\lambda(u_2) = R(u_1 \wedge u_2) = 0,$$

and we conclude that the  $A^\lambda(u_1)$ 's (and hence the  $A^\lambda(u_2)$ 's) lie in the subspace of  $V$  generated by the  $A^\lambda(u)$ 's; in fact, by Cartan's lemma,

$$A^\lambda(u_1) = \sum_{\mu=n+1}^N c_\mu^\lambda A^\mu(u), \quad c_\mu^\lambda = c_\lambda^\mu,$$

where the  $c_\mu^\lambda$ 's are real numbers. Similarly we can show that the  $A^\lambda(v_1)$ 's and the  $A^\lambda(v_2)$ 's lie in the span of the  $A^\lambda(v)$ 's. After a possible change of orthonormal basis for the normal space we can arrange that

$$(4) \quad A^\lambda(u_1) = c_\lambda A^\lambda(u), \quad A^\lambda(u_2) = (1 - c_\lambda) A^\lambda(u),$$

where  $c_\lambda = 1$  for  $n+1 \leq \lambda \leq q$ ,  $c_\lambda \neq 0, 1$  for  $q+1 \leq \lambda \leq r$ , and  $c_\lambda = 0$  for  $r+1 \leq \lambda \leq N$ . Equations (1), (2), and (4) now imply that

$$\begin{aligned} \sum_{\lambda=n+1}^r c_\lambda A^\lambda(u) \wedge A^\lambda(v_2) &= \sum_{\lambda=n+1}^N A^\lambda(u_1) \wedge A^\lambda(v_2) = R(u_1 \wedge v_2) = 0, \\ \sum_{\lambda=q+1}^N A^\lambda(v_1) \wedge (1 - c_\lambda) A^\lambda(u) &= 0. \end{aligned}$$

Therefore  $A^{n+1}(v_2), \dots, A^r(v_2)$  and  $A^{q+1}(v_1), \dots, A^N(v_1)$  are in the span of the  $A^\lambda(u)$ 's. But they are also in the span of the  $A^\lambda(v)$ 's, and since the  $A^\lambda(u)$ 's and the  $A^\lambda(v)$ 's are linearly independent we must have

$$(5) \quad A^\lambda(v_2) = 0 \text{ for } n+1 \leq \lambda \leq r, \quad A^\lambda(v_1) = 0 \text{ for } q+1 \leq \lambda \leq N.$$

In particular the vectors  $A^\lambda(v)$ ,  $q+1 \leq \lambda \leq r$ , must vanish, and since these vectors are linearly independent,  $q=r$ .

We adopt the following index conventions:  $n+1 \leq \alpha \leq q$ ,  $q+1 \leq \rho \leq N$ . Then  $c_\alpha=1$ ,  $c_\rho=0$ , and it follows from (4) and (5) that

$$A^\alpha(u_1) = A^\alpha(u), \quad A^\alpha(v_1) = A^\alpha(v), \quad A^\rho(u_1) = A^\rho(v_1) = 0.$$

Hence if  $w_2 \in V_2$ ,

$$\begin{aligned} \sum_{\alpha} A^\alpha(u) \wedge A^\alpha(w_2) &= \sum_{\lambda=n+1}^N A^\lambda(u_1) \wedge A^\lambda(w_2) = R(u_1 \wedge w_2) = 0, \\ \sum_{\alpha} A^\alpha(v) \wedge A^\alpha(w_2) &= 0. \end{aligned}$$

Since the  $2(q-n)$  vectors  $A^\alpha(u)$ ,  $A^\alpha(v)$  are linearly independent, we can conclude that  $A^\alpha(w_2)=0$ . Similarly we can show that  $A^\rho(w_1)=0$  when  $w_1 \in V_1$ . Therefore

$$\Phi^\alpha(w_1, w_2) = \langle w_1, A^\alpha(w_2) \rangle = 0, \quad \Phi^\rho(w_1, w_2) = \langle A^\rho(w_1), w_2 \rangle = 0,$$

for  $w_1 \in V_1$ ,  $w_2 \in V_2$ . This establishes (3) at  $(m_1, m_2)$  and finishes the proof of the Reducibility Theorem.

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