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COMPLEMENTING SETS OF n-TUPLES OF INTEGERS

MELVYN B. NATHANSON[1]

ABSTRACT. Let $S, A_1, A_2, \cdots, A_p$ be finite nonempty sets of n-tuples of integers such that, if $a_i \in A_i$ for $i=1, 2, \cdots, p$, then $a_1+a_2+\cdots+a_p \in S$, and such that every $s \in S$ has a unique representation as a sum $s=a_1+a_2+\cdots+a_p$ with $a_i \in A_i$. If S is the cartesian product of n sets of integers, then each A_i is also the cartesian product of n sets of integers, and conversely.

Let $S, A_1, A_2, \cdots, A_p$ be sets of n-tuples of integers. Define addition of n-tuples componentwise. Then S is the *sum* of $A_1, A_2, \cdots, A_p$, denoted $S=A_1+A_2+\cdots+A_p$, if $S=\{a_1+a_2+\cdots+a_p | a_i \in A_i \text{ for } i=1, 2, \cdots, p\}$. If S is the sum of $A_1, A_2, \cdots, A_p$, and if for each $s \in S$ there exist unique n-tuples $a_i \in A_i$ such that $s=a_1+a_2+\cdots+a_p$, then $A_1, A_2, \cdots, A_p$ are called *complementing sets* for S, denoted $S \cong A_1+A_2+\cdots+A_p$. A set of n-tuples is *proper* if it is the cartesian product of n sets of integers. For positive integers u and v, let $S=\{0, 1, 2, \cdots, u\} \times \{0, 1, 2, \cdots, v\}$. If A_1 and A_2 are subsets of S such that $S \cong A_1+A_2$, then Hansen [1] proved that A_1 and A_2 are proper. This result extends to the general case of arbitrary n and p, and S the cartesian product of any n finite sets of integers.

THEOREM. *Let $S, A_1, A_2, \cdots, A_p$ be finite nonempty sets of n-tuples of integers such that $S \cong A_1+A_2+\cdots+A_p$. Then S is proper if and only if each A_i is proper.*

PROOF. Suppose that S is proper. By translation, it is enough to consider only the case in which all coordinates of all n-tuples of $S, A_1, \cdots, A_p$ are nonnegative integers. Let $Z[X_1, \cdots, X_n]$ be the ring of polynomials in n variables with integral coefficients. Define $F, G_1, G_2, \cdots, G_p$ in $Z[X_1, \cdots, X_n]$ by

$$F = \sum_{(s_1, s_2, \cdots, s_n) \in S} X_1^{s_1} X_2^{s_2} \cdots X_n^{s_n},$$

$$G_i = \sum_{(a_1, a_2, \cdots, a_n) \in A_i} X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}.$$

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Since S is proper, there exist finite sets of nonnegative integers $S_1, S_2, \cdots, S_n$ such that $S=S_1\times S_2\times\cdots\times S_n$. Then

$$F=\prod_{j=1}^{n}\Big(\sum_{s_j\in S_j} X_j^{s_j}\Big). \tag{1}$$

The polynomial F is the product of irreducible polynomials, and by (1), each of these irreducibles is a polynomial in exactly one variable. Each G_i is also a product of irreducible polynomials. Since $A_1+A_2+\cdots+A_p\cong S$, it follows that $F=G_1G_2\cdots G_p$. Since $Z[X_1, \cdots, X_n]$ is a unique factorization domain, each irreducible factor of each G_i is a factor of F, and so is a polynomial in exactly one variable. Thus there exist polynomials $g_{ij}\in Z[X_j]$ for $i=1, 2, \cdots, p$ and $j=1, 2, \cdots, n$ such that $G_i=g_{i1}g_{i2}\cdots g_{in}$. Let A_{ij} be the finite set of nonnegative integers which are the powers of X_j occurring with nonzero coefficient in g_{ij}. Then $A_i=A_{i1}\times A_{i2}\times\cdots\times A_{in}$ for $i=1, 2, \cdots, p$, and A_i is proper. (Moreover, $A_{1j}+A_{2j}+\cdots+A_{pj}\cong S_j$ for $j=1, 2, \cdots, n$.)

Conversely, suppose that each A_i is proper. Then there exist finite sets of integers A_{ij} for $i=1, 2, \cdots, p$ and $j=1, 2, \cdots, n$ such that $A_i=A_{i1}\times A_{i2}\times\cdots\times A_{in}$. Let $S_j=A_{1j}+A_{2j}+\cdots+A_{pj}$ for $j=1, 2, \cdots, n$. Then $S=S_1\times S_2\times\cdots\times S_n$, and so S is proper.

REMARK. The theorem is false if S is the cartesian product of infinite sets. If N is the set of nonnegative integers, there exist sets A_1 and A_2 which are not proper but satisfy $A_1+A_2\cong N\times N$. Hansen [1] and Niven [2] have determined all sets A_1 and A_2 such that $A_1+A_2\cong N\times N$.

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UNIVERSITY OF ROCHESTER, ROCHESTER, NEW YORK 14627

Current address: Southern Illinois University, Carbondale, Illinois 62901