COMPLEMENTING SETS OF $n$-TUPLES OF INTEGERS

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Abstract. Let $S, A_1, A_2, \ldots, A_p$ be finite nonempty sets of $n$-tuples of integers such that, if $a_i \in A_i$ for $i=1, 2, \ldots, p$, then $a_1 + a_2 + \cdots + a_p \in S$, and such that every $s \in S$ has a unique representation as a sum $s = a_1 + a_2 + \cdots + a_p$ with $a_i \in A_i$. If $S$ is the cartesian product of $n$ sets of integers, then each $A_i$ is also the cartesian product of $n$ sets of integers, and conversely.

Let $S, A_1, A_2, \ldots, A_p$ be sets of $n$-tuples of integers. Define addition of $n$-tuples componentwise. Then $S$ is the sum of $A_1, A_2, \ldots, A_p$, denoted $S = A_1 + A_2 + \cdots + A_p$, if $S = \{a_1 + a_2 + \cdots + a_p | a_i \in A_i$ for $i=1, 2, \ldots, p \}$. If $S$ is the sum of $A_1, A_2, \ldots, A_p$, and if for each $s \in S$ there exist unique $n$-tuples $a_i \in A_i$ such that $s = a_1 + a_2 + \cdots + a_p$, then $A_1, A_2, \ldots, A_p$ are called complementing sets for $S$, denoted $S \subseteq A_1 + A_2 + \cdots + A_p$. A set of $n$-tuples is proper if it is the cartesian product of $n$ sets of integers. For positive integers $u$ and $v$, let $S = \{0, 1, 2, \ldots, u\} \times \{0, 1, 2, \ldots, v\}$. If $A_1$ and $A_2$ are subsets of $S$ such that $S \subseteq A_1 + A_2$, then Hansen [1] proved that $A_1$ and $A_2$ are proper. This result extends to the general case of arbitrary $n$ and $p$, and $S$ the cartesian product of any $n$ finite sets of integers.

Theorem. Let $S, A_1, A_2, \ldots, A_p$ be finite nonempty sets of $n$-tuples of integers such that $S \subseteq A_1 + A_2 + \cdots + A_p$. Then $S$ is proper if and only if each $A_i$ is proper.

Proof. Suppose that $S$ is proper. By translation, it is enough to consider only the case in which all coordinates of all $n$-tuples of $S, A_1, \ldots, A_p$ are nonnegative integers. Let $Z[X_1, \ldots, X_n]$ be the ring of polynomials in $n$ variables with integral coefficients. Define $F, G_1, G_2, \ldots, G_p$ in $Z[X_1, \ldots, X_n]$ by

$$F = \sum_{(a_1, a_2, \ldots, a_p) \in S} X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n},$$

$$G_i = \sum_{(a_1, a_2, \ldots, a_n) \in A_i} X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}.$$
Since $S$ is proper, there exist finite sets of nonnegative integers $S_1, S_2, \ldots, S_n$ such that $S = S_1 \times S_2 \times \cdots \times S_n$. Then

\begin{equation}
F = \prod_{j=1}^{n} \left( \sum_{i_{j} \in S_{j}} X_{i}^{j} \right).
\end{equation}

The polynomial $F$ is the product of irreducible polynomials, and by (1), each of these irreducibles is a polynomial in exactly one variable. Each $G_i$ is also a product of irreducible polynomials. Since $A_1 + A_2 + \cdots + A_p \cong S$, it follows that $F = G_1 G_2 \cdots G_p$. Since $\mathbb{Z}[X_1, \ldots, X_n]$ is a unique factorization domain, each irreducible factor of each $G_i$ is a factor of $F$, and so is a polynomial in exactly one variable. Thus there exist polynomials $g_{ij} \in \mathbb{Z}[X_j]$ for $i=1, 2, \ldots, p$ and $j=1, 2, \ldots, n$ such that $G_i = g_{i1} g_{i2} \cdots g_{in}$. Let $A_{ij}$ be the finite set of nonnegative integers which are the powers of $X_j$ occurring with nonzero coefficient in $g_{ij}$. Then $A_i = A_{i1} \times A_{i2} \times \cdots \times A_{in}$ for $i=1, 2, \ldots, p$, and $A_i$ is proper. (Moreover, $A_{i1} + A_{i2} + \cdots + A_{ip} \cong S_i$ for $j=1, 2, \ldots, n$.)

Conversely, suppose that each $A_i$ is proper. Then there exist finite sets of integers $A_{ij}$ for $i=1, 2, \ldots, p$ and $j=1, 2, \ldots, n$ such that $A_i = A_{i1} \times A_{i2} \times \cdots \times A_{in}$. Let $S_j = A_{1j} + A_{2j} + \cdots + A_{pj}$ for $j=1, 2, \ldots, n$. Then $S = S_1 \times S_2 \times \cdots \times S_n$, and so $S$ is proper.

**Remark.** The theorem is false if $S$ is the cartesian product of infinite sets. If $\mathbb{N}$ is the set of nonnegative integers, there exist sets $A_1$ and $A_2$ which are not proper but satisfy $A_1 + A_2 \cong \mathbb{N} \times \mathbb{N}$. Hansen [1] and Niven [2] have determined all sets $A_1$ and $A_2$ such that $A_1 + A_2 \cong \mathbb{N} \times \mathbb{N}$.

**References**


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