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ON THE NOETHERIAN-LIKE RINGS OF E. G. EVANS

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ABSTRACT. It is shown that if a commutative ring with identity R is nonnoetherian, then the polynomial ring in one indeterminate over R has an ideal with infinitely many maximal prime divisors (in the sense of Nagata).

Let R denote a commutative ring with 1, and for any ideal A of R, let $\mathscr{Z}(A)=\{r\in R \mid \text{there exists } s\in R\backslash A \text{ such that } rs\in A\}$. (By "ideal" we shall always mean ideal$\neq R$. The notation $R\backslash A$ denotes the set-complement of A in R.) $\mathscr{Z}(A)$ is merely the set of zero-divisors on the R-module R/A and is always a union of prime ideals of R. Evans [1] calls R a ZD-ring (zero-divisor ring) if for any ideal A of R, $\mathscr{Z}(A)$ is a union of finitely many prime ideals. We shall prove here the following:

THEOREM. *R is noetherian if (and only if) the polynomial ring in one indeterminate $R[X]$ is a ZD-ring.*

Evans has proved in [1] the 2 indeterminate analogue of this theorem (which follows from the theorem) and the special case of the theorem for R containing an infinite field.

A prime ideal P of R such that P is maximal with respect to the property of being contained in $\mathscr{Z}(A)$ is called a maximal N-prime (for Nagata-prime) of A. Note that such a prime contains A and that $\mathscr{Z}(A)$ is the union of the maximal N-primes of A. (See [2] and [4] for a perspective on the associated primes of an ideal.)

PROOF OF THEOREM. Suppose R is not noetherian. Then there exists a strictly ascending chain $(0)<(a_1)<(a_1, a_2)<\cdots<(a_1, \cdots, a_n)<\cdots$ of ideals of R. Let $f_0=X, f_1=1+X, \cdots, f_i=1+f_0f_1\cdots f_{i-1}, \cdots$. We wish to show that the ideal $A=(a_1f_1, a_2f_1f_2, \cdots, a_nf_1\cdots f_n, \cdots)$ in $R[X]$ has an infinite number of maximal N-primes and hence has the property that $\mathscr{Z}(A)$ is not a finite union of prime ideals. We show first that each $f_i\in\mathscr{Z}(A)$. Since $A\subset(f_1)$ and f_1 is a monic polynomial of positive degree in $R[X]$, it follows that $A\cap R=(0)$. Hence $a_1\notin A$, so $a_1f_1\in A$ implies that $f_1\in\mathscr{Z}(A)$. Similarly, to show $f_n\in\mathscr{Z}(A)$, we wish to show $a_nf_1\cdots f_{n-1}\notin A$. Consider

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the residue class ring $R/(a_1, \cdots, a_{n-1})=R'$. The image of the ideal A in $R'[X]$ is generated by the elements $a'_n f'_1 \cdots f'_n$, $a'_{n+1} f'_1 \cdots f'_{n+1}, \cdots$, where " ′ " denotes image in $R'[X]$. It will suffice to show that

$$a'_n f'_1 \cdots f'_{n-1} \notin (a'_n f'_1 \cdots f'_n, a'_{n+1} f'_1 \cdots f'_{n+1}, \cdots);$$

and since $f'_1 \cdots f'_{n-1}$ is a monic polynomial in $R'[X]$, this is equivalent to showing that $a'_n \notin (a'_n f'_n, a'_{n+1} f'_n f'_{n+1}, \cdots) \subset (f'_n)$. Since f'_n is a monic polynomial of positive degree in $R'[X]$, we have $(f'_n) \cap R' = (0)$. Thus $a'_n \notin (a'_n f'_n, a'_{n+1} f'_n f'_{n+1}, \cdots)$; hence we have proved $f_n \in \mathscr{Z}(A)$.

Consider now f_i and f_j for $i \neq j$. Clearly no prime ideal of $R[X]$ contains both f_i and f_j. Since each f_i is in $\mathscr{Z}(A)$ and hence is in some maximal N-prime of A, it then follows that A has infinitely many maximal N-primes. Q.E.D.[2]

A Lasker ring is one for which every ideal is a finite intersection of primary ideals. Such rings have been studied by Krull [**3**], and Evans has observed that every Lasker ring is a ZD-ring. Thus a consequence of the above theorem is that R is noetherian if (and only if) $R[X]$ is Lasker.

We can add one further bit of information on the relationship between the ZD and noetherian properties.

PROPOSITION. *If R is a ZD-ring and R_P is noetherian for every prime ideal P of R, then R is noetherian.*

PROOF. By [**2**, Corollary 1.4] it suffices to show that every ideal A of R has only finitely many B_w-primes (a B_w-prime, or weak-Bourbaki prime, of A is a prime ideal P such that P is a minimal prime divisor of $A:x$ for some $x \in R$). If P is a B_w-prime of A, then P is contained in a maximal N-prime of A; and since R is a ZD-ring, A has only a finite number of maximal N-primes, say $Q_1, \cdots, Q_n$. Moreover, P is a B_w-prime of A in R and $P \subset Q_i$ imply PR_{Q_i} is a B_w-prime of AR_{Q_i} [**2**, Proposition 1.2]. Since R_{Q_i} is noetherian, AR_{Q_i} has only a finite number of B_w-primes. Hence A can have only finitely many B_w-primes.

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