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NONUNIQUENESS OF COEFFICIENT RINGS IN A POLYNOMIAL RING

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ABSTRACT. An example is given of commutative rings B, C with 1 such that $B\not\cong C$ but $B[t]\cong C[t]$, where t is an indeterminate.

Several authors [1], [2], [3] have recently studied the question, if $B[t]\cong C[t]$ (B, C are commutative rings with 1, t is an indeterminate), does $B\cong C$ follow? A simple counterexample is given below.

Let R be the reals and let P, Q, t, U, V, W, X, Y, Z be indeterminates. Let $A=R[X, Y, Z]/(X^2+Y^2+Z^2-1)=R[x, y, z]$. Let $\phi:A^3\rightarrow A$ by $\phi(a, b, c)=ax+by+cz$. Then ϕ splits: map a to $a(x, y, z)$. $E=\ker\phi$ is well known to be a rank 2 projective which is not free, and hence requires 3 generators (that E is not free may be deduced from the fact that the tangent bundle of the real 2-sphere has no nonvanishing continuous sections). The splitting of ϕ shows that $A^3\cong E\oplus A$. If we pass to symmetric algebras, we obtain the isomorphisms

$$S(A^3)\cong A[P, Q, t]\cong S(E)\otimes_A S(A)\cong S(E)\otimes_A A[t]\cong S(E)[t],$$

and since $E\cong A^3/(x, y, z)A$,

$$S(E)\cong A[U, V, W]/(xU+yV+zW).$$

Let $B=A[P, Q]$ and $C=A[U, V, W]/(xU+yV+zW)$. We have shown that $B[t]\cong C[t]$. It remains only to show that $B\not\cong C$. Suppose $h:B\cong C$. B and C are A-subalgebras of the polynomial ring $B[t]=A[P, Q, t]$ over A. It is easy to show that the only invertible elements of A, hence of $B[t]$, and therefore of B and C, are the nonzero real numbers. Since R has no nontrivial automorphisms, h must be an R-isomorphism. It is easy to check that A is a formally real domain. If D is a formally real domain and T is an indeterminate over D, the only solutions of $X^2+Y^2+Z^2=1$ in $D[T]$ already lie in D. Hence, the only solutions of this equation in $B[t]$ lie in A, and the same holds for B and C. Thus, $h(A)\subset A$, and $h^{-1}(A)\subset A$. After composing h with the automorphism of B which agrees with h^{-1} on A and fixes P, Q, we can assume that h is an A-isomorphism of B and C. C is a

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graded A-algebra. It follows that there are two elements $c=c_0+c_1+\cdots$, $c'=c'_0+c'_1+\cdots$ (where c_i or c'_i is the i-form component of c or c') such that $C=A[c, c']=A[c-c_0, c'-c'_0]$. It follows easily that c_1, c'_1 span the A-module of 1-forms of C. But this module is isomorphic to E, and E requires three generators, a contradiction. Thus, $B\not\cong C$.

A similiar example has been noted by M. P. Murthy (unpublished).

Bibliography

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