PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, July 1972

NONUNIQUENESS OF COEFFICIENT RINGS IN A POLYNOMIAL RING

M. HOCHSTER[1]

ABSTRACT. An example is given of commutative rings B, C with 1 such that $B \not\cong C$ but $B[t] \cong C[t]$, where t is an indeterminate.

Several authors [1], [2], [3] have recently studied the question, if $B[t] \cong C[t]$ (B, C are commutative rings with 1, t is an indeterminate), does $B \cong C$ follow? A simple counterexample is given below.

Let R be the reals and let P, Q, t, U, V, W, X, Y, Z be indeterminates. Let $A = R[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1) = R[x, y, z]$. Let $\phi : A^3 \to A$ by $\phi(a, b, c) = ax + by + cz$. Then ϕ splits: map a to $a(x, y, z)$. $E = \ker \phi$ is well known to be a rank 2 projective which is not free, and hence requires 3 generators (that E is not free may be deduced from the fact that the tangent bundle of the real 2-sphere has no nonvanishing continuous sections). The splitting of ϕ shows that $A^3 \cong E \oplus A$. If we pass to symmetric algebras, we obtain the isomorphisms

$$S(A^3) \cong A[P, Q, t] \cong S(E) \otimes_A S(A) \cong S(E) \otimes_A A[t] \cong S(E)[t],$$

and since $E \cong A^3/(x, y, z)A$,

$$S(E) \cong A[U, V, W]/(xU + yV + zW).$$

Let $B = A[P, Q]$ and $C = A[U, V, W]/(xU + yV + zW)$. We have shown that $B[t] \cong C[t]$. It remains only to show that $B \not\cong C$. Suppose $h : B \cong C$. B and C are A-subalgebras of the polynomial ring $B[t] = A[P, Q, t]$ over A. It is easy to show that the only invertible elements of A, hence of $B[t]$, and therefore of B and C, are the nonzero real numbers. Since R has no nontrivial automorphisms, h must be an R-isomorphism. It is easy to check that A is a formally real domain. If D is a formally real domain and T is an indeterminate over D, the only solutions of $X^2 + Y^2 + Z^2 = 1$ in $D[T]$ already lie in D. Hence, the only solutions of this equation in $B[t]$ lie in A, and the same holds for B and C. Thus, $h(A) \subset A$, and $h^{-1}(A) \subset A$. After composing h with the automorphism of B which agrees with h^{-1} on A and fixes P, Q, we can assume that h is an A-isomorphism of B and C. C is a

Received by the editors October 13, 1971.
AMS 1970 *subject classifications*. Primary 13B25.
[1] Research supported in part by NSF grant GP-29224X.

graded A-algebra. It follows that there are two elements $c=c_0+c_1+\cdots$, $c'=c'_0+c'_1+\cdots$ (where c_i or c'_i is the i-form component of c or c') such that $C=A[c, c']=A[c-c_0, c'-c'_0]$. It follows easily that c_1, c'_1 span the A-module of 1-forms of C. But this module is isomorphic to E, and E requires three generators, a contradiction. Thus, $B\not\cong C$.

A similiar example has been noted by M. P. Murthy (unpublished).

Bibliography

1. S. Abhyankar, P. Eakin and W. Heinzer, *On the uniqueness of the coefficient ring in a polynomial ring* (to appear).

2. D. Coleman and E. Enochs, *Polynomial invariance of rings*, Proc. Amer. Math. Soc. **25** (1970), 559–562.

3. P. Eakin and K. K. Kubota, *A note on the uniqueness of rings of coefficients in polynomial rings* (to appear).

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455