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THE ZETA FUNCTION OF TORAL ENDOMORPHISMS

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ABSTRACT. In this note we give a completely elementary way to compute the number of fixed points of a certain class of toral endomorphisms. This, in turn, gives the zeta function of these endomorphisms.

We first review the small collection of facts needed to understand the theorem. We denote by $\boldsymbol{R}^n$ the group of n-tuples of real numbers and by Z^n the subgroup of $\boldsymbol{R}^n$ of integral lattice points. The n-torus may be thought of as the group obtained by factoring Z^n out of $\boldsymbol{R}^n$, i.e. $\boldsymbol{R}^n/Z^n$. If A is an $n\times n$ matrix with all integer entries then, since A leaves Z^n invariant, it induces an endomorphism T_A on $\boldsymbol{R}^n/Z^n$ by $T_A(x+Z^n)=Ax+Z^n$. It can be shown that if T is any endomorphism of $\boldsymbol{R}^n/Z^n$ then there exists a matrix A with all integer entries such that $T=T_A$. It is easy to see that if $T_A^k=T_A\cdots T_A$, k terms, then $T_A^k=T_{A^k}$. The class of endomorphisms we wish to consider are those whose corresponding matrices have no eigenvalues equal to a root of unity. In case $\det A=\pm 1$ then these are exactly the ergodic automorphisms [**1**].

THEOREM. *Let A be a $n\times n$ integral entry matrix with the property that it has no eigenvalues equal to a root of unity. Let T_A denote the endomorphism that A induces on the n-torus, $\boldsymbol{R}^n/Z^n$. If N_k denotes the number of fixed points of $T_A^k=T_{A^k}$ then $N_k=|\det(A^k-I)|$ for $k=1,2,3,\cdots$.*

PROOF. For $\tilde{x}\in\boldsymbol{R}^n/Z^n$ we write $\tilde{x}=x+Z^n$ with $x=(x_1,\cdots,x_n)$, $0\leqq x_i<1$, $i=1,2,\cdots,n$. Let k, a positive integer, be given. Now, $\tilde{x}$ is fixed under T_{A^k} if and only if $A^kx=x \bmod (Z^n)$, or, if and only if $(A^k-I)x\in Z^n$. Since A^k has no eigenvalues equal to 1 it follows that A^k-I is one-to-one. Thus there is a one-to-one correspondence between these points x of the half open unit cube which have the property that $(A^k-I)x\in Z^n$, and the integral lattice points contained in the image of the half open unit cube under A^k-I. Considering the volume interpretation of determinates it is geometrically clear that this number is exactly

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$|\det(A^k - I)|$. A rather dull proof of this fact can be seen as follows. If we let $(A^k - I) \mid Z^n$ denote the restriction of $(A^k - I)$ to Z^n, then the number of integral lattice points in the image of the half open unit cube under $A^k - I$ is equal to the order of the group $Z^n/\mathrm{Im}[(A^k - I) \mid Z^n]$. Now, if $p = |\det(A^k - I)|$ then p is an integer and for $x \in Z^n$, $x \in \mathrm{Im}[(A^k - I) \mid Z^n]$ if and only if $x = py$ for some $y \in Z^n$. This proves the theorem.

Recall from [**2**, p. 764] that if $f: X \to X$ is continuous and if N_k equals the number of isolated fixed points of f^k and if $N_k < \infty$ for all k then the Artin-Mazur zeta function of f is defined to be

$$\zeta_f(z) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k}{k} z^k \right).$$

The above theorem gives the zeta function for all toral endomorphisms for which $N_k < \infty$ for all k, that is, for all toral endomorphisms which have no eigenvalues equal to a root of unity. The result in this case is the same as the result in the case when A has no eigenvalues on the unit circle which is given in [**2**, p. 769].

Peter Walters has recently shown me that, independently, he has obtained the same proof.

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