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ON THE LOCALIZATION OF RECTANGULAR PARTIAL SUMS FOR MULTIPLE FOURIER SERIES

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ABSTRACT. The question of the localization for rectangular partial sums of the multiple Fourier series for functions of Sobolev spaces is settled.

1. **Introduction.** As usual we denote by T_n the n-dimensional torus $(-\pi, +\pi]\times\cdots\times(-\pi, +\pi]$, by $\dot{W}_p^1(\mathring{T}_n)=W_p^1$ the Sobolev space of functions which are absolutely continuous and periodic with period 2π on almost all those lines which are perpendicular to the hyperfaces of T_n with the superscript and the subscript having their usual meanings. Furthermore, we use $\mathring{W}_p^1(\mathring{T}_n)=\mathring{W}_p^1$ to denote the space of those functions of $\dot{W}_p^1$ which vanish on the boundary of T_n. Naturally, we overlook the difference between a function and the class of functions for which it is a representative, and for convenience we always choose the representation functions as described above.

Goffman and Liu have established in [2] that the square partial sum has the localization property for $\dot{W}_p^1$ if $p\geqq n-1$ that for each $p<n-1$ there is an $f\in \dot{W}_p^1$ which does not have the localization property, and that there is an everywhere differentiable function on T_2 for which the localization property fails. It is also shown in [2] that the rectangular partial sum does not have the localization property for the space $\dot{W}_p^1$ if $p=n-1$. Our purpose in this note is to show that the localization property does hold for the rectangular partial sum if $p>n-1$ and therefore settle completely the question of localization for the rectangular partial sums of Fourier series so far as the Sobolev space $\dot{W}_p^1$ is concerned. Regarding almost everywhere convergence for rectangular sums, it was shown by Cesari [1], in contrast to our results, that almost everywhere convergence holds for $\dot{W}_p^1$, for $n=2$, and for $\dot{W}_p^1, p>1$, for $n>2$. It seems accordingly that Cesari's work deserves more attention than it has received.

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For our purpose, we consider in §2 the estimates of the Dirichlet integrals of Lip. α functions on T_n over subintervals of T_n. As a consequence we obtain the uniform convergence for the rectangular partial sums of functions in $\dot{W}_p^1$, $p>n$. It has been shown in [7] that the Dini-Lipschitz theorem holds on T_2 for the rectangular partial sum. Since the method of the proof in [7] is unnecessarily complicated and it is not clear that the method employed there can be applied to the higher dimensional cases, we will indicate at the appropriate place that the corresponding theorem in T_n actually follows along the lines of the arguments in §2.

2. **Dirichlet integrals and uniform convergence.** For convenience we shall use the following notations: Capital letters X, Y, $\cdots$ are points in R^k, $k=1, 2, \cdots$, small letters x, y, $\cdots$ are real numbers; if $X=(x_1, \cdots, x_n)$, then $X_j=(x_1, \cdots, x_j)$, $\bar{X}_j=(x_{j+1}, \cdots, x_n)$ and $dX=dx_1 \cdots dx_n=(dx_1 \cdots dx_j)(dx_{j+1} \cdots dx_n)=dX_j\, d\bar{X}_j$; if $J=(j_1, \cdots, j_k)$ is a k-dimensional lattice point with positive components, then $j_J=\max\{j_1, \cdots, j_k\}$, and $D_J(Y)=D_{j_1}(y_1)\cdot D_{j_2}(y_2) \cdots D_{j_k}(y_k)$ is the corresponding multiple Dirichlet kernel, where $D_j(y)=\pi^{-1}\{\sin(j+\frac{1}{2})y/2 \sin \frac{1}{2}y\}$.

THEOREM 1. *If f is a* Lip. α *function on T_n, $\alpha>0$, and if*

$$I = [-a_1, +a_1] \times [-a_2, +a_2] \times \cdots \times [-a_n, +a_n] \subset T_n,$$

then

$$\left| \int \{f(X + Y) - f(X)\} D_j(Y)\, dY \right| \leqq C\, |f|_{L,\alpha} \sum_{k=0}^{n-1} (j_{\bar{J}_k})^{-\alpha} \cdot \prod_{l=k+1}^{n} (\log j_l),$$

where $|f|_{L,\alpha}$ is the Lip. α *norm of f on T_n and C is a constant which depends only on $a_1, \cdots, a_n$.*

PROOF. Write

$$\begin{aligned} f(X+Y) - f(X) &= \sum_{k=0}^{n-1} \{f(X_k, (X_k + Y_k)^{-}) - f(X_{k+1}, (X_{k+1} + Y_{k+1})^{-})\} \\ &\equiv \sum_{k=0}^{n-1} \phi_k(X, Y), \end{aligned}$$

where $(X_0, (X_0+Y_0)^{-})=X+Y$ and $(X_n, (X_n+Y_n)^{-})=X$.

Obviously, by rearranging the variables if necessary, we may assume without loss of generality that $j_1 \geqq \cdots \geqq j_n$ i.e. $j_{\bar{J}_k}=j_{k+1}$, $k=0, \cdots, n-1$.

Now

$$\left| \int_I \{f(X + Y) - f(X)\}\, D_J(Y) \right| \leqq \sum_{k=0}^{n-1} \left| \int_I \phi_k(X, Y)\, D_J(Y)\, dY \right| \equiv \sum_{k=0}^{n-1} R_k.$$

$$
\begin{aligned}
R_k &= \left| \int_I \{ f(X_k, x_{k+1} + y_{k+1}, (X_{k+1} + Y_{k+1})^{\frown}) \right. \\
&\qquad \left. - f(X_k, x_{k+1}, (X_{k+1} + Y_{k+1})^{\frown}) \} D_J(Y)\, dY \right| \\
&= \left| \int_{-a_n}^{a_n} \cdots \int_{-a_{k+2}}^{a_{k+2}} \left| \int_{-a_{k+1}}^{a_{k+1}} \{ f(X_k, x_{k+1} + y_{k+1}, (X_{k+1} + Y_{k+1})^{\frown}) \right. \right. \\
&\qquad - f(X_k, x_{k+1}, (X_{k+1} Y_{k+1})^{\frown}) \} D_{j_{k+1}}(y_{k+1})\, dy_{k+1} \\
&\qquad \left. \left. \times \int_{-a_k}^{a_k} \cdots \int_{-a_1}^{a_1} D_{J_k}(Y_k)\, dY_k \right| D_{J_{k+1}}(Y_{k+1})\, dY_{k+1} \right| \\
&\leqq A \left| \int_{-a_n}^{a_n} \cdots \int_{-a_{k+2}}^{a_{k+2}} \left[\int_{-a_{k+1}}^{a_{k+1}} \{ f(X_k, x_{k+1} + y_{k+1}, (X_{k+1} + Y_{k+1})^{\frown}) \right. \right. \\
&\qquad \left. - f(X_k, x_{k+1}, (X_{k+1} + Y_{k+1})^{\frown}) \} D_{j_{k+1}}(y_{k+1})\, dy_{k+1} \right] \\
&\qquad \left. \times D_{\bar{J}_{k+1}}(\bar{Y}_{k+1})\, d\bar{Y}_{k+1} \right| \\
&\leqq A' \, |f|_{L,\alpha} j_{k+1}^{-\alpha} \log j_{k+1} \int_{-a_n}^{a_n} \cdots \int_{-a_{k+2}}^{a_{k+2}} |D_{\bar{J}_{k+1}}(\bar{Y}_{k+1})| \, d\bar{Y}_{k+1} \\
&\leqq C \, |f|_{L,\alpha} (j_{\bar{J}_k})^{-\alpha} \prod_{l=k+1}^{n} (\log j_l),
\end{aligned}
$$

where the last two steps are familiar in the 1-dimensional case (see [**8**, pp. 62–64]). Q.E.D.

THEOREM 2. *If $f \in \dot{W}_p^1$, $p > n$, then the rectangular partial sums of the Fourier series of f converge uniformly to f on T_n.*

PROOF. It is known that if $f \in \dot{W}_p^1$, $p > n$, then f is a Lip. $(1-(n/p))$ function (see [**5**, p. 83]). Therefore Theorem 2 follows from Theorem 1 with $a_1 = \cdots = a_n = \pi$. Q.E.D.

COROLLARY. *If $f \in \mathring{W}_p^l$ and $p > n/l$, then the rectangular partial sums of the Fourier series of f converges uniformly to f on T_n.*

PROOF. By a well-known lemma of Sobolev (see [**6**] or [**5**]) $f \in \mathring{W}_q^1$, $q > n$, if $f \in \mathring{W}_p^l$, $p > n/l$. Therefore the corollary follows readily from Theorem 2.

For results which are similar to the corollary for the spherical summation method see [**3**] and [**4**].

As far as uniform convergence is concerned, it is clear that the following theorem which is the n-dimensional analogue of the Dini-Lipschitz theorem can be proved along the lines of arguments in the proof for Theorem 1.

THEOREM 3. *Let f be continuous and periodic on T_n and let $w(t)$ be the modulus of continuity of f. If $w(t)=o(\log (1/t))^{-n}$, then the rectangular partial sums of the Fourier series of f converge uniformly to f on T_n.*

3. **Localization.** Now let us turn to the questions of localization. In view of the application to convergence questions we put the localization principle in the following form

THEOREM 4. *Let $T_n^b=\{X\in T_n:\max\{|x_1|,\cdots,|x_n|\}\geqq b\}$, $0<b<\pi$. If $f\in \dot{W}_p^1$, $p>n-1$, then*

$$\lim_{j_1,\cdots,j_n\to\infty}\int_{T_n^b} f(X+Y)\,D_J(Y)\,dY=0$$

uniformly in $X\in T_n$.

PROOF. First of all, if $f\in \dot{W}_p^1$, then for almost all y_i, $g_{y_i}(Y_{i-1},\bar{Y}_i)=f(Y_{i-1},y_i,\bar{Y}_i)$ is a function in $\dot{W}_p^1(T_{n-1})$, $i=1,\cdots,n$. For these y_i, if $p>n-1$, g_{y_i} is a Lip. $(1-(n-1)/p)$ function on T_{n-1} with its Lip. $(1-(n-1)/p)$ norm bounded by $C\,\|g_{y_i}\|^1_{p,(n-1)}$, where $\|g_{y_i}\|^1_{p,(n-1)}$ is the $\dot{W}_p^1$-norm of g_{y_i} on T_{n-1} and C is a constant which depends only on P, $(n-1)$, and T_{n-1} (see [5, p. 83]).

Next,

$$\begin{aligned}\int_{T_n^b} f(X+Y)\,D_J(Y)\,dY&=\int_{T_{n-1}}\int_{|y_1|>b} f(X+Y)\,D_J(Y)\,dY\\ (*)\qquad&+\int_{T_{n-2}}\int_{|y_2|\geqq b}\int_{|y_1|<b} f(X+Y)\,D_J(Y)\,dY\\ &+\cdots+\int_{|y_n|\geqq b}\int_{|y_{n-1}|<b}\cdots\int_{|y_1|<b} f(X+Y)\,D_J(Y)\,dY.\end{aligned}$$

We need only estimate the first term on the right-hand side of (*), the estimates for the other terms being similar. In the following we shall use J' for $\bar{J}_1$ and write $J'=(j_1',j_2',\cdots,j_{n-1}')$ where $j_k'=j_{k+1}$, $k=1,\cdots,n-1$.

$$\begin{aligned}\int_{T_{n-1}}\int_{|y_1|\geqq b}& f(X+Y)\,D_J(Y)\,dY\\ &=\int_{T_{n-1}}\int_{|y_1|\geqq b}\{f(X+Y)-f(x_1+y_1,\bar{X}_1)\}\,D_J(Y)\,dY\\ &\quad+\int_{T_{n-1}}\int_{|y_1|\geqq b} f(x_1+y_1,\bar{X}_1)\,D_J(Y)\,dY\\ &\equiv R_1+R_2,\end{aligned}$$

where

$$(\Delta_1)\qquad R_1 = \int_{|y_1|\geqq b}\int_{T_{n-1}} \{f(x_1+y_1,(X_1+Y_1)^{-}) - f(x_1+y_1,\bar{X}_1)\} \times D_{J'}(\bar{Y}_1)\,dY_1\,D_{j_1}(y_1)\,dy_1,$$

and

$$(\Delta_2)\qquad R_2 = \int_{|y_1|\geqq b} f(x_1+y_1,\bar{X}_1)\,D_{j_1}(y_1)\,dy_1.$$

In view of the remarks in the first paragraph of the proof and by applying Theorem 1 with n replaced by $(n-1)$ we have, for almost all y_1,

$$\begin{aligned}
&\left|\int_{T_{n-1}} \{f(x_1+y_1,X_1+Y_1) - f(x_1+y_1,\bar{X}_1)\}D_{J'}(\bar{Y}_1)\,d\bar{Y}_1\right| \\
&\quad = \left|\int_{T_{n-1}} \{g_{x_1+y_1}(X_1+Y_1) - g_{x_1+y_1}(\bar{X}_1)\}D_{J'}(\bar{Y}_1)\,d\bar{Y}_1\right| \\
&\quad \leqq C\|g_{x_1+y_1}\|^1_{P,(n-1)} \sum_{k=0}^{n-2} (j_{J_{k'}})^{(n-1)/P}\cdot \prod_{l=k+1}^{n-1} (\log j_l'),
\end{aligned}$$

where C is a constant depending only on P, $(n-1)$, and T_{n-1}.

Consequently, from (Δ_1) we have

$$|R_1| \leqq \frac{C}{b}\left\{\sum_{k=0}^{n-2} (j_{J_k})^{(n-1)/P}\cdot \prod_{l=k+1}^{n-1} \log j_l'\right\}\cdot (2\pi)^{(p-1)/P}\cdot \|f\|^1_{P,n},$$

and therefore $R_1 \to 0$ uniformly in X as $j_1,\cdots,j_n\to\infty$.

Finally, we show that $R_2\to 0$ uniformly in X as $j_1\to\infty$. Let $\varepsilon>0$ be given, choose $\delta>0$ such that

$$(\Delta_3)\qquad C\cdot(2\pi)^{(P-1)/P}\cdot \|f\|^1_{P,n}\cdot \delta^{1-(n-1)/P}\cdot b^{-1} < \varepsilon/2,$$

where C is the constant chosen previously in the proof. Let $Z_1, Z_2, \cdots, Z_N$ be points of T_{n-1} such that any point of T_{n-1} will be within the δ-neighborhood of at least one of $Z_1, Z_2, \cdots, Z_N$ and such that

$$(\Delta_4)\qquad \lim_{j_1\to\infty} \int_{|y_1|\geqq b} f(x_1+y_1, Z_l)\,D_{j_1}(y_1)\,dy_1 = 0$$

uniformly in x_1 for $l=1,\cdots,N$. That this can be done is obvious from the Fubini theorem and the 1-dimensional localization principle. Now let X be any point of T_n. There is l with $1\leqq l\leqq N$ such that

$$(\Delta_5)\qquad |Z_l - \bar{X}_1| < \delta$$

where by $|X|$ we mean the euclidean norm of X in the corresponding appropriate space. Write

$$(\Delta_6)\qquad \begin{aligned} R_2 &= \int_{|y_1|\geqq b} \{f(x_1+y_1, \bar{X}_1) - f(x_1+y_1, Z_l)\}\, D_{j_1}(y_1)\, dy_1 \\ &\quad + \int_{|y_1|\geqq b} f(x_1+y_1, Z_l)\, D_{j_1}(y_1)\, dy_1 \\ &\equiv R_2' + R_2''. \end{aligned}$$

That $|R_2''| < \varepsilon/2$ if j_1 is sufficiently large and independent of X follows from (Δ_4). We shall have shown that $|R_2| < \varepsilon$ for sufficiently large j_1 and independent of X if we show that $|R_2'| < \varepsilon/2$ for all j_1. As pointed out in the first paragraph of the proof, for almost all y_1, the following inequality holds

$$|f(x_1+y_1, \bar{X}_1) - f(x_1+y_1, Z_l)| \leqq C\, \|g_{x_1+y_1}\|^1_{P,(n-1)}\, |\bar{X}_1 - Z_l|^{1-(n-1)/P};$$

therefore

$$\begin{aligned} |R_2'| &\leqq \int_{|y_1|\geqq b} |f(x_1+y_1, \bar{X}_1) - f(x_1+y_1, Z_l)| \cdot |D_{j_1}(y_1)|\, dy_1 \\ &\leqq \frac{1}{b} \int_{-\pi}^{\pi} |f(x_1+y_1, \bar{X}_1) - f(x_1+y_1, Z_l)|\, dy_1 \\ &\leqq \frac{C}{b} (2\pi)^{(P-1)/P} \cdot \|f\|^1_{P,n} \cdot |\bar{X}_1 - Z|^{1-(n-1)/P} < \frac{\varepsilon}{2}, \end{aligned}$$

by (Δ_3), (Δ_5), and the Hölder inequality. Q.E.D.

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