PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, July 1972

LINEAR OPERATORS FOR WHICH T^*T AND TT^* COMMUTE[1]

STEPHEN L. CAMPBELL

ABSTRACT. Linear operators T for which T^*T and TT^* commute are studied. Examples are given to show that this class of operators is distinct from several other operator classes. It is proven that if T^*T and TT^* commute and T is hyponormal, then T has an invariant subspace. A generalization of this theorem is given.

In this paper we shall investigate bounded linear operators T on a separable Hilbert space H for which T^*T and TT^* commute. After establishing the basic properties of such operators, we will give examples and discuss how this class of operators relates to those already studied by previous authors. Finally we shall show that if such an operator is hyponormal then it has an invariant subspace.

1. In studying a general operator T, the operators T^*T and TT^* frequently occur. For example they appear both in the polar decomposition of an operator and in the models for contractions [**2**, §6]. A natural question to ask then is for which operators do T^*T and TT^* commute?

If T_1 and T_2 are linear operators, then by definition $[T_1, T_2]=T_1T_2-T_2T_1$. We will call an operator binormal if $[T^*T, TT^*]=0$. Clearly the set of all binormal operators is closed in the uniform operator topology and contains the quasinormal operators. T is quasinormal if $[T, T^*T]=0$.

2. Our first theorem gives a simple characterization of binormal operators.

THEOREM 1. *If T has the polar decomposition $T=UP$ where the null space of U equals the null space of P, then* $[T^*T, TT^*]=0$ *if and only if* $[P, UPU^*]=0$.

Received by the editors July 22, 1971.

AMS 1970 *subject classifications*. Primary 47A15, 47B99; Secondary 47B20.

Key words and phrases. Hyponormal operators, invariant subspaces, operators for which T^*T and TT^* commute.

[1] This paper is a portion of the author's doctoral dissertation being written under the direction of Professor Allen Devinatz for Northwestern University.

This research was partially supported by National Science Foundation grant GP-27950.

PROOF. We observe first that U^*U is the projection onto the range of P. Thus $U^*UP=P$ since $P\geqq 0$. Then $[T^*T, TT^*]=0 \Leftrightarrow [P^2, UP^2U^*]=0 \Leftrightarrow [P, (UP^2U^*)^{1/2}]=0 \Leftrightarrow [P, UPU^*]=0$.

3. The set of all binormal operators, (BN), has some nice algebraic properties as the next easily proved proposition shows.

PROPOSITION 1. *If* $T\in(\mathrm{BN})=\{T:[T^*T, TT^*]=0\}$ *and if* α *is any complex number, then*

(1) $\alpha T\in(\mathrm{BN})$,

(2) $T^*\in(\mathrm{BN})$, *and*

(3) $T^{-1}\in(\mathrm{BN})$ *if it exists.*

PROOF. The proof is obvious.

However, (BN) is not closed under addition even when the operators commute. For let $T_1=\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$ and $T_2=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$. It is easy to verify that $T_1\in(\mathrm{BN})$. Clearly $T_2\in(\mathrm{BN})$. Let $T=T_1+T_2=\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$. Then $[T^*T, TT^*]=\left[\begin{smallmatrix}0&-2\\2&0\end{smallmatrix}\right]$ and $T\notin(\mathrm{BN})$. Notice that this example also shows that $T\in(\mathrm{BN})$ does not imply $\mu I+T\in(\mathrm{BN})$ for complex scalars μ.

4. We will now show that (BN) is independent of several major classes of operators.

DEFINITION 1. T is hyponormal if and only if $T^*T-TT^*\geqq 0$.

DEFINITION 2. T is normaloid if and only if the spectral radius of T, $r(T)$, equals $\|T\|$.

DEFINITION 3. T is spectraloid if and only if the numerical radius of T, $w(T)$, equals $r(T)$.

It is well known that if T is hyponormal, then it is normaloid. Furthermore, normaloid implies spectraloid. If T is in any of these three classes, then T need not satisfy the equation $[T^*T, TT^*]=0$.

EXAMPLE 1. If $T=\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$, then $\|T\|=1$, $w(T)=\frac{1}{2}$, $r(T)=0$, and T is binormal. Thus there are nonspectraloid operators in (BN).

EXAMPLE 2. Let S be a weighted shift. Then S^*S and SS^* are both diagonal operators and hence commute. In particular, let T be the unilateral weighted shift with weight sequence $\{1, \frac{1}{2}, 1, 1, \cdots\}$. Then $r(T)=w(T)=\|T\|=1$. T is not hyponormal since the weight sequence is not monotonic nondecreasing. Thus T is binormal and normaloid but not hyponormal.

EXAMPLE 3. If T is the bilateral weighted shift $Te_i=ae_{i+1}$ for $i\leqq 0$, $Te_1=be_2$, $Te_i=e_{i+1}$ for $i\geqq 2$, $0<a<b<1$, and $\{e_i\}_{i=-\infty}^{\infty}$ an orthonormal basis for H, then T is hyponormal but not subnormal [**3**, Problem 160].

REMARK. Examples 1, 2, and 3 are not new but illustrate the diversity of binormal operators. Since the shifts are all binormal, it is easy to construct further examples such as compact operators.

That the examples are all shifts or shiftlike (nilpotent) is due entirely to ease of construction. There are many other binormal operators. For example $T=\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$ is binormal and is not normal and not nilpotent. In fact, $T^2=I$.

5. We note that the operator in Example 3 is hyponormal and invertible but not subnormal. Thus the next theorem involves a nontrivial class of operators.

THEOREM 2. *If T is hyponormal and $[T^*T, TT^*]=0$, then T has an invariant subspace.*

PROOF. T trivially has an invariant subspace unless both T and T^* are one-to-one. Let $A=TT^*$ and $B=T^*T$. $B\geqq A\geqq 0$ since T is hyponormal.

If (a, b) is an open interval and $E_C(\delta)$ is the spectral measure for the selfadjoint operator C, then

$$E_C(a, b) = \lim_{\delta\to 0^+} \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{a+\delta}^{b-\delta} g(C, \mu, \varepsilon)\, d\mu$$

where

$$\begin{aligned} g(C, \mu, \varepsilon) &= (1/2i)\{((\mu - \varepsilon i) - C)^{-1} - ((\mu + \varepsilon i) - C)^{-1}\} \\ &= \varepsilon((\mu - C)^2 + \varepsilon^2)^{-1} \end{aligned}$$

and the limits are strong limits [**1**, p. 921]. Since $TB=AT$ we have $TE_B(a, b)=E_A(a, b)T$.

Let $F(\delta)$ be the spectral measure of the commutative Banach algebra generated by A and B. $A=\int f(s)\, F(ds)$ and $B=\int g(s)\, F(ds)$ for some measurable functions f, g [**1**, p. 895]. $g(s)\geqq f(s)\geqq 0$ almost everywhere dF since $B\geqq A\geqq 0$. Also $\sigma(A)$=essential range of f and $\sigma(B)$=the essential range of g. $f(s)>0$ almost everywhere dF since A is one-to-one. Let $X_\delta(\cdot)$ be the characteristic function of the set δ.

Then

$$E_B(0, b) = X_{(0,b)}(B) = \int X_{(0,b)}(g(s))\, F(ds) = F(\{s: 0 < g(s) < b\}).$$

Similarly $E_A(0, b)=F(\{s:0<f(s)<b\})$. $E_B(0, b)\leqq E_A(0, b)$ since

$$\{s: 0 < g(s) < b\} \subseteq \{s: 0 < f(s) < b\}$$

except for a set of zero dF measure. But $TE_B(0, b)=E_A(0, b)T$. Let N_C be the null space of $E_C(0, b)$. Then $TN_B\subseteq N_A\subseteq N_B$ and T will have an invariant subspace provided we can find a b for which $E_B(0, b)$ is nontrivial. Such a b exists unless B has a one point spectrum. But then $T^*T=\alpha I$, $\alpha>0$, and $\alpha^{-1/2}T$ is an isometry. Then $\alpha^{-1/2}T$ has an invariant subspace. Hence T does.

REMARK. If T is binormal, then so is T^*. We could have assumed then in Theorem 2 that T or T^* were hyponormal, that is, that T was seminormal.

6. Theorem 2 is a special case of the next theorem.

THEOREM 3. *Let $A=TT^*$, $B=T^*T$, and $E_C(\delta)$ the spectral measure of C for $C=A$ or $C=B$ and δ a Borel subset of the real line. If there is an interval (a, b) such that $E_A(a, b)$ and $E_B(a, b)$ are nontrivial projections and $E_A(a, b)\geqq E_B(a, b)$ or $E_B(a, b)\geqq E_A(a, b)$, then T has an invariant subspace.*

PROOF. As in the proof of Theorem 2 we may assume that T is one-to-one with dense range. Furthermore $TE_B(a, b)=E_A(a, b)T$ for any interval (a, b). Suppose (a, b) is as stated in Theorem 3. If $E_A(a, b)\geqq E_B(a, b)$, we have $N_A\subseteq N_B$. Thus $TN_B\subseteq N_A\subseteq N_B$ and T has an invariant subspace. If $E_B(a, b)\geqq E_A(a, b)$, let $R_C=N_C^\perp$. Then $TR_B\subseteq R_A\subseteq R_B$ and T again has an invariant subspace.

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators*. II: *Spectral theory. Selfadjoint operators in Hilbert space*, Interscience, New York, 1963. MR **32** #6181.

2. P. A. Fillmore, *Notes on operator theory*, Van Nostrand Reinhold Math. Studies, no. 30, Van Nostrand, Reinhold, New York, 1970, MR **41** #2414.

3. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N.J., 1967. MR **34** #8178.

DEPARTMENT OF MATHEMATICS, MARQUETTE UNIVERSITY, MILWAUKEE, WISCONSIN 53233 (Current address of Stephen L. Campbell)

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201