LINEAR OPERATORS FOR WHICH $T^*T$ AND $TT^*$ COMMUTE

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Abstract. Linear operators $T$ for which $T^*T$ and $TT^*$ commute are studied. Examples are given to show that this class of operators is distinct from several other operator classes. It is proven that if $T^*T$ and $TT^*$ commute and $T$ is hyponormal, then $T$ has an invariant subspace. A generalization of this theorem is given.

In this paper we shall investigate bounded linear operators $T$ on a separable Hilbert space $H$ for which $T^*T$ and $TT^*$ commute. After establishing the basic properties of such operators, we will give examples and discuss how this class of operators relates to those already studied by previous authors. Finally we shall show that if such an operator is hyponormal then it has an invariant subspace.

1. In studying a general operator $T$, the operators $T^*T$ and $TT^*$ frequently occur. For example they appear both in the polar decomposition of an operator and in the models for contractions [2, §6]. A natural question to ask then is for which operators do $T^*T$ and $TT^*$ commute?

If $T_1$ and $T_2$ are linear operators, then by definition $[T_1, T_2] = T_1T_2 - T_2T_1$. We will call an operator binormal if $[T^*T, TT^*] = 0$. Clearly the set of all binormal operators is closed in the uniform operator topology and contains the quasinormal operators. $T$ is quasinormal if $[T, T^*T] = 0$.

2. Our first theorem gives a simple characterization of binormal operators.

Theorem 1. If $T$ has the polar decomposition $T = UP$ where the null space of $U$ equals the null space of $P$, then $[T^*T, TT^*] = 0$ if and only if $[P, UPU^*] = 0$.
Proof. We observe first that $U^*U$ is the projection onto the range of $P$. Thus $U^*UP = P$ since $P \geq 0$. Then $[T^*T, TT^*] = 0 \iff [P^2, UPU^*] = 0 \iff [P, (UP^2U^*)^{1/2}] = 0 \Rightarrow [P, UPU^*] = 0$.

3. The set of all binormal operators, $(BN)$, has some nice algebraic properties as the next easily proved proposition shows.

**Proposition 1.** If $T \in (BN) \equiv \{T : [T^*T, TT^*] = 0\}$ and if $\alpha$ is any complex number, then

1. $\alpha T \in (BN)$,
2. $T^* \in (BN)$, and
3. $T^{-1} \in (BN)$ if it exists.

**Proof.** The proof is obvious.

However, $(BN)$ is not closed under addition even when the operators commute. For let $T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. It is easy to verify that $T_1 \in (BN)$. Clearly $T_2 \in (BN)$. Let $T = T_1 + T_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then $[T^*T, TT^*] = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ and $T \notin (BN)$. Notice that this example also shows that $T \in (BN)$ does not imply $\mu I + T \in (BN)$ for complex scalars $\mu$.

4. We will now show that $(BN)$ is independent of several major classes of operators.

**Definition 1.** $T$ is hyponormal if and only if $T^*T - TT^* \geq 0$.

**Definition 2.** $T$ is normaloid if and only if the spectral radius of $T$, $r(T)$, equals $\|T\|$.

**Definition 3.** $T$ is spectraloid if and only if the numerical radius of $T$, $w(T)$, equals $r(T)$.

It is well known that if $T$ is hyponormal, then it is normaloid. Furthermore, normaloid implies spectraloid. If $T$ is in any of these three classes, then $T$ need not satisfy the equation $[T^*T, TT^*] = 0$.

**Example 1.** If $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then $\|T\| = 1$, $w(T) = \frac{1}{2}$, $r(T) = 0$, and $T$ is binormal. Thus there are nonspectraloid operators in $(BN)$.

**Example 2.** Let $S$ be a weighted shift. Then $S^*S$ and $SS^*$ are both diagonal operators and hence commute. In particular, let $T$ be the unilateral weighted shift with weight sequence $\{1, \frac{1}{3}, 1, 1, \ldots\}$. Then $r(T) = w(T) = \|T\| = 1$. $T$ is not hyponormal since the weight sequence is not monotonic nondecreasing. Thus $T$ is binormal and normaloid but not hyponormal.

**Example 3.** If $T$ is the bilateral weighted shift $Te_i = ae_{i+1}$ for $i \leq 0$, $Te_i = be_i$ for $i \geq 2$, $0 < a < b < 1$, and $\{e_i\}^-\infty_{i=-\infty}^\infty$ an orthonormal basis for $H$, then $T$ is hyponormal but not subnormal [3, Problem 160].

**Remark.** Examples 1, 2, and 3 are not new but illustrate the diversity of binormal operators. Since the shifts are all binormal, it is easy to construct further examples such as compact operators.
That the examples are all shifts or shiftlike (nilpotent) is due entirely to ease of construction. There are many other binormal operators. For example \( T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) is binormal and is not normal and not nilpotent. In fact, \( T^2 = I \).

5. We note that the operator in Example 3 is hyponormal and invertible but not subnormal. Thus the next theorem involves a nontrivial class of operators.

**Theorem 2.** If \( T \) is hyponormal and \([T^*T, TT^*] = 0\), then \( T \) has an invariant subspace.

**Proof.** \( T \) trivially has an invariant subspace unless both \( T \) and \( T^* \) are one-to-one. Let \( A = TT^* \) and \( B = T^*T \). \( B \geq A \geq 0 \) since \( T \) is hyponormal.

If \((a, b)\) is an open interval and \( E_C(\delta) \) is the spectral measure for the selfadjoint operator \( C \), then

\[
E_C(a, b) = \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{a+\delta}^{b-\delta} g(C, \mu, \varepsilon) \, d\mu
\]

where

\[
g(C, \mu, \varepsilon) = \frac{1}{(2\pi)^n} \left( ((\mu - \varepsilon i) - C)^{-1} - ((\mu + \varepsilon i) - C)^{-1} \right)
\]

and the limits are strong limits [1, p. 921]. Since \( TB = AT \) we have \( TEB(a, b) = E_A(a, b)T \).

Let \( F(\delta) \) be the spectral measure of the commutative Banach algebra generated by \( A \) and \( B \). \( A = \int f(s) F(ds) \) and \( B = \int g(s) F(ds) \) for some measurable functions \( f, g \) [1, p. 895]. \( g(s) \geq f(s) \geq 0 \) almost everywhere \( dF \) since \( B \geq A \geq 0 \). Also \( \sigma(A) = \text{essential range of } f \) and \( \sigma(B) = \text{the essential range of } g \). \( f(s) > 0 \) almost everywhere \( dF \) since \( A \) is one-to-one. Let \( X_\delta(\cdot) \) be the characteristic function of the set \( \delta \).

Then

\[
E_B(0, b) = X_{(0,b)}(B) = \int X_{(0,b)}(g(s)) F(ds) = F(\{s : 0 < g(s) < b\}).
\]

Similarly \( E_A(0, b) = F(\{s : 0 < f(s) < b\}) \). \( E_B(0, b) \leq E_A(0, b) \) since \( \{s : 0 < g(s) < b\} \subseteq \{s : 0 < f(s) < b\} \) except for a set of zero \( dF \) measure. But \( TEB(0, b) = E_A(0, b)T \). Let \( N_C \) be the null space of \( E_C(0, b) \). Then \( TN_B \subseteq N_A \subseteq N_B \) and \( T \) will have an invariant subspace provided we can find a \( b \) for which \( E_A(0, b) \) is nontrivial. Such a \( b \) exists unless \( B \) has a one point spectrum. But then \( T^*T = \alpha I, \alpha > 0 \), and \( \alpha^{-1/2}T \) is an isometry. Then \( \alpha^{-1/2}T \) has an invariant subspace. Hence \( T \) does.
Remark. If $T$ is binormal, then so is $T^*$. We could have assumed then in Theorem 2 that $T$ or $T^*$ were hyponormal, that is, that $T$ was semi-normal.

6. Theorem 2 is a special case of the next theorem.

Theorem 3. Let $A = TT^*$, $B = T^*T$, and $E_C(\delta)$ the spectral measure of $C$ for $C = A$ or $C = B$ and $\delta$ a Borel subset of the real line. If there is an interval $(a, b)$ such that $E_A(a, b)$ and $E_B(a, b)$ are nontrivial projections and $E_A(a, b) \supseteq E_B(a, b)$ or $E_B(a, b) \supseteq E_A(a, b)$, then $T$ has an invariant subspace.

Proof. As in the proof of Theorem 2 we may assume that $T$ is one-to-one with dense range. Furthermore $TE_B(a, b) = E_A(a, b)T$ for any interval $(a, b)$. Suppose $(a, b)$ is as stated in Theorem 3. If $E_A(a, b) \supseteq E_B(a, b)$, we have $N_A \subseteq N_B$. Thus $TN_B \subseteq N_A \subseteq N_B$ and $T$ has an invariant subspace. If $E_B(a, b) \supseteq E_A(a, b)$, let $R_C = N_C^\perp$. Then $TR_B \subseteq R_A \subseteq R_B$ and $T$ again has an invariant subspace.

References


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