ON AN EXPLICIT CHARACTERIZATION OF SPHERICAL CURVES

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Abstract. It will be proved that the "explicit characterization" of spherical curves recently obtained by S. Breuer and D. Gottlieb (Proc. Amer. Math. Soc. 27 (1971), pp. 126-127) is, without any precondition on the curvature and torsion, a necessary and sufficient condition for a curve to be a spherical curve. The proof is based on an earlier result of the present author on spherical curves (Monatsh. Math. 67 (1963), pp. 363-365).

1. Introduction. The curves to be considered here are curves in a Euclidean 3-space with an equation of the form $x=x(s)$, $s \in [0, L]$, where $s$ is the arc length and the vector function $x(s)$ is of class $C^4$. For such a curve, the following facts are well known:

(i) The nonnegative curvature ($C^2$-function) $k_1$ defined on $[0, L]$ is unique.

(ii) If $k_1$ is nowhere zero, the torsion ($C^1$-function) $k_2$ defined on $[0, L]$ is unique except for a sign.

(iii) These two functions $k_1$, $k_2$ completely determine the shape and size of the curve.

In books on elementary differential geometry (see for example [2, p. 32]), the condition for a curve to be a spherical curve, i.e. for it to lie on a sphere, is usually given in the form

$$[k_2^{-1}(k_1^{-1})']' + k_2k_1^{-1} = 0,$$

where the prime denotes differentiation with respect to the arc length $s$.

Clearly, condition (1.1) has a meaning only if $k_1$ and $k_2$ are nowhere zero, and it is only under this precondition that (1.1) is a necessary and sufficient condition for a curve to be a spherical curve. When "cleared of fractions", (1.1) takes the form

$$(-k_1k_1'' + 2k_1^2)k_2 + k_1'k_2' + k_1^2k_2^2 = 0.$$
Although condition (1.2) has a meaning whether or not $k_1$ and $k_2$ are nowhere zero, it is neither a necessary condition nor a sufficient condition for a curve to be a spherical curve unless $k_1$ and $k_2$ are nowhere zero [5, Theorem 5.2].

A condition for a curve to be a spherical curve which is both necessary and sufficient and which holds without any precondition on $k_1$ or $k_2$ has been given by the author [4] in the following form:

**Theorem 1.1.** A $C^4$-curve $x=x(s)$, $s \in [0, L]$, parametrized by its arc length $s$, is a spherical curve if and only if

(i) its curvature $k_1$ is nowhere zero (then its torsion $k_2$ is defined), and

(ii) there exists a $C^1$-function $f$ defined on $[0, L]$ such that

\[ fk_2 = (k_1^{-1})', \quad f' + k_2 k_1^{-3} = 0. \]

Moreover, a curve satisfying this condition lies on a sphere of radius $(k_1^{-2} + f^2)^{1/2}$ (which is of course a constant).

Quite recently, S. Breuer and D. Gottlieb [1], using a result of theirs on differential equations, derived from (1.1) the following “explicit characterization” of spherical curves:

\[ k_1^{-1}(s) = A \cos \int_0^s k_2 \, ds + B \sin \int_0^s k_2 \, ds, \]

where $A$ and $B$ are constants. Since (1.4) is deduced from (1.1), we can hardly expect that it would be a necessary and a sufficient condition for a curve to be a spherical curve without the precondition that $k_1$ and $k_2$ are nowhere zero. It is therefore a pleasant surprise that this should turn out to be the case, as we shall show by proving the following theorem.

**Theorem 1.2.** A $C^4$-curve $x=x(s)$, $s \in [0, L]$, parametrized by its arc length $s$, with curvature $k_1$ and torsion $k_2$ is a spherical curve if and only if

\[ \left( A \cos \int_0^s k_2 \, ds + B \sin \int_0^s k_2 \, ds \right) k_1(s) = 1, \]

where $A$, $B$ are constants. Moreover, a curve satisfying condition (1.5) lies on a sphere of radius $(A^2 + B^2)^{1/2}$.

**2. Proof of Theorem 1.2.**

**Sufficiency.** It suffices to show that if a curve satisfies condition (1.5), then it satisfies the conditions in Theorem 1.1. First, (1.5) implies that $k_1$ is nowhere zero. Next, writing (1.5) as

\[ A \cos \int_0^s k_2 \, ds + B \sin \int_0^s k_2 \, ds = k_1^{-1}(s) \]
and differentiating, we get
\[ \left( -A \sin \int_0^s k_2 \, ds + B \cos \int_0^s k_2 \, ds \right) k_2(s) = (k_1^{-1})'(s). \]

It can now be verified that the function \( f \) defined by
\[ f(s) = -A \sin \int_0^s k_2 \, ds + B \cos \int_0^s k_2 \, ds \]
satisfies condition (1.3). This completes the proof of the sufficiency of condition (1.5).

**Necessity.** Suppose that \( x=x(s), \ s \in [0, L], \) is a spherical curve. Then the conditions in Theorem 1.1 are satisfied. Let us define a \( C^2 \)-function \( \theta \) and two \( C^1 \)-functions \( g \) and \( h \) on \([0, L]\) by

\[ (2.1) \quad \theta(s) = \int_0^s k_2(s) \, ds, \]
\[ (2.2) \quad g(s) = k_1^{-1}(s) \cos \theta(s) - f(s) \sin \theta(s), \]
\[ h(s) = k_1^{-1}(s) \sin \theta(s) + f(s) \cos \theta(s). \]

If we differentiate equations (2.2) with respect to \( s \) and take account of (2.1) and (1.3), we find that \( g' \) and \( h' \) are both identically zero. Therefore, \( g(s) = A, \ h(s) = B, \) where \( A, \ B \) are constants. Now substituting these in (2.2) and solving the resulting equations for \( k_1^{-1}(s) \), we get
\[ k_1^{-1}(s) = A \cos \theta(s) + B \sin \theta(s), \]
which is (1.5). This proves the necessity of condition (1.5).

Finally, to prove the last assertion in Theorem 1.2, we note from the first part of this section that, for a curve satisfying the condition \( k_1^{-1} = A \cos \theta + B \sin \theta \), the function \( f = -A \sin \theta + B \cos \theta \) satisfies condition (1.3) in Theorem 1.1. Therefore, by Theorem 1.1, this curve lies on a sphere with radius
\[ [(k_1^{-1})^2 + f^2]^{1/2} = (A^2 + B^2)^{1/2}. \]

The proof of Theorem 1.2 is now complete.

3. **A remark.** Condition (1.4) was originally derived from (1.1) by S. Breuer and D. Gottlieb [1] by using a rather profound result of theirs on differential equations. But it can also be derived very simply as follows.

As in (2.1), we let \( \theta(s) = \int_0^s k_2 \, ds \). Then the differential equation (1.1) can be rewritten as
\[ (3.1) \quad d^2k_1^{-1}/d\theta^2 + k_1^{-1} = 0, \]
the solution of which is $k_1^{-1} = A \cos \theta + B \sin \theta$, where $A$, $B$ are constants. This proves (1.4).

It is interesting to note that C. E. Weatherburn [3, p. 25, Exercise 3] came very close to obtaining (1.4), but he stopped at (3.1).

REFERENCES


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