

## ON AN EXPLICIT CHARACTERIZATION OF SPHERICAL CURVES

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**ABSTRACT.** It will be proved that the "explicit characterization" of spherical curves recently obtained by S. Breuer and D. Gottlieb (Proc. Amer. Math. Soc. **27** (1971), pp. 126–127) is, without any precondition on the curvature and torsion, a necessary and sufficient condition for a curve to be a spherical curve. The proof is based on an earlier result of the present author on spherical curves (Monatsh. Math. **67** (1963), pp. 363–365).

**1. Introduction.** The curves to be considered here are curves in a Euclidean 3-space with an equation of the form  $\mathbf{x}=\mathbf{x}(s)$ ,  $s \in [0, L]$ , where  $s$  is the arc length and the vector function  $\mathbf{x}(s)$  is of class  $C^4$ . For such a curve, the following facts are well known:

(i) The nonnegative curvature ( $C^2$ -function)  $k_1$  defined on  $[0, L]$  is unique.

(ii) If  $k_1$  is nowhere zero, the torsion ( $C^1$ -function)  $k_2$  defined on  $[0, L]$  is unique except for a sign.

(iii) These two functions  $k_1, k_2$  completely determine the shape and size of the curve.

In books on elementary differential geometry (see for example [2, p. 32]), the condition for a curve to be a spherical curve, i.e. for it to lie on a sphere, is usually given in the form

$$(1.1) \quad [k_2^{-1}(k_1^{-1})']' + k_2 k_1^{-1} = 0,$$

where the prime denotes differentiation with respect to the arc length  $s$ .

Clearly, condition (1.1) has a meaning only if  $k_1$  and  $k_2$  are nowhere zero, and it is only under this precondition that (1.1) is a necessary and sufficient condition for a curve to be a spherical curve. When "cleared of fractions", (1.1) takes the form

$$(1.2) \quad (-k_1 k_1'' + 2k_1'^2)k_2 + k_1 k_1' k_2' + k_1^2 k_2^3 = 0.$$

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Although condition (1.2) has a meaning whether or not  $k_1$  and  $k_2$  are nowhere zero, it is neither a necessary condition nor a sufficient condition for a curve to be a spherical curve unless  $k_1$  and  $k_2$  are nowhere zero [5, Theorem 5.2].

A condition for a curve to be a spherical curve which is both necessary and sufficient and which holds without any precondition on  $k_1$  or  $k_2$  has been given by the author [4] in the following form:

**THEOREM 1.1.** *A  $C^4$ -curve  $\mathbf{x}=\mathbf{x}(s)$ ,  $s \in [0, L]$ , parametrized by its arc length  $s$ , is a spherical curve if and only if*

- (i) *its curvature  $k_1$  is nowhere zero (then its torsion  $k_2$  is defined), and*
- (ii) *there exists a  $C^1$ -function  $f$  defined on  $[0, L]$  such that*

$$(1.3) \quad f k_2 = (k_1^{-1})', \quad f' + k_2 k_1^{-1} = 0.$$

*Moreover, a curve satisfying this condition lies on a sphere of radius  $(k_1^{-2} + f^2)^{1/2}$  (which is of course a constant).*

Quite recently, S. Breuer and D. Gottlieb [1], using a result of theirs on differential equations, derived from (1.1) the following "explicit characterization" of spherical curves:

$$(1.4) \quad k_1^{-1}(s) = A \cos \int_0^s k_2 ds + B \sin \int_0^s k_2 ds,$$

where  $A$  and  $B$  are constants. Since (1.4) is deduced from (1.1), we can hardly expect that it would be a necessary and a sufficient condition for a curve to be a spherical curve without the precondition that  $k_1$  and  $k_2$  are nowhere zero. It is therefore a pleasant surprise that this should turn out to be the case, as we shall show by proving the following theorem.

**THEOREM 1.2.** *A  $C^4$ -curve  $\mathbf{x}=\mathbf{x}(s)$ ,  $s \in [0, L]$ , parametrized by its arc length  $s$ , with curvature  $k_1$  and torsion  $k_2$  is a spherical curve if and only if*

$$(1.5) \quad \left( A \cos \int_0^s k_2 ds + B \sin \int_0^s k_2 ds \right) k_1(s) = 1,$$

*where  $A, B$  are constants. Moreover, a curve satisfying condition (1.5) lies on a sphere of radius  $(A^2 + B^2)^{1/2}$ .*

## 2. Proof of Theorem 1.2.

**SUFFICIENCY.** It suffices to show that if a curve satisfies condition (1.5), then it satisfies the conditions in Theorem 1.1. First, (1.5) implies that  $k_1$  is nowhere zero. Next, writing (1.5) as

$$A \cos \int_0^s k_2 ds + B \sin \int_0^s k_2 ds = k_1^{-1}(s)$$

and differentiating, we get

$$\left(-A \sin \int_0^s k_2 ds + B \cos \int_0^s k_2 ds\right) k_2(s) = (k_1^{-1})'(s).$$

It can now be verified that the function  $f$  defined by

$$f(s) = -A \sin \int_0^s k_2 ds + B \cos \int_0^s k_2 ds$$

satisfies condition (1.3). This completes the proof of the sufficiency of condition (1.5).

NECESSITY. Suppose that  $x=x(s)$ ,  $s \in [0, L]$ , is a spherical curve. Then the conditions in Theorem 1.1 are satisfied. Let us define a  $C^2$ -function  $\theta$  and two  $C^1$ -functions  $g$  and  $h$  on  $[0, L]$  by

$$(2.1) \quad \theta(s) = \int_0^s k_2(s) ds,$$

$$(2.2) \quad \begin{aligned} g(s) &= k_1^{-1}(s) \cos \theta(s) - f(s) \sin \theta(s), \\ h(s) &= k_1^{-1}(s) \sin \theta(s) + f(s) \cos \theta(s). \end{aligned}$$

If we differentiate equations (2.2) with respect to  $s$  and take account of (2.1) and (1.3), we find that  $g'$  and  $h'$  are both identically zero. Therefore,  $g(s)=A$ ,  $h(s)=B$ , where  $A$ ,  $B$  are constants. Now substituting these in (2.2) and solving the resulting equations for  $k_1^{-1}(s)$ , we get

$$k_1^{-1}(s) = A \cos \theta(s) + B \sin \theta(s),$$

which is (1.5). This proves the necessity of condition (1.5).

Finally, to prove the last assertion in Theorem 1.2, we note from the first part of this section that, for a curve satisfying the condition  $k_1^{-1}=A \cos \theta+B \sin \theta$ , the function  $f=-A \sin \theta+B \cos \theta$  satisfies condition (1.3) in Theorem 1.1. Therefore, by Theorem 1.1, this curve lies on a sphere with radius

$$[(k_1^{-1})^2 + f^2]^{1/2} = (A^2 + B^2)^{1/2}.$$

The proof of Theorem 1.2 is now complete.

**3. A remark.** Condition (1.4) was originally derived from (1.1) by S. Breuer and D. Gottlieb [1] by using a rather profound result of theirs on differential equations. But it can also be derived very simply as follows.

As in (2.1), we let  $\theta(s)=\int_0^s k_2 ds$ . Then the differential equation (1.1) can be rewritten as

$$(3.1) \quad d^2 k_1^{-1} / d\theta^2 + k_1^{-1} = 0,$$

the solution of which is  $k_1^{-1} = A \cos \theta + B \sin \theta$ , where  $A, B$  are constants. This proves (1.4).

It is interesting to note that C. E. Weatherburn [3, p. 25, Exercise 3] came very close to obtaining (1.4), but he stopped at (3.1).

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