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A CHARACTERIZATION OF HEREDITARILY INDECOMPOSABLE CONTINUA

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ABSTRACT. Hereditarily indecomposable continua are characterized by giving a condition which some defining sequence for the continua must satisfy.

In [1], Lida K. Barrett gave a necessary and sufficient condition, in terms of chains, for a chainable continuum to be indecomposable. W. T. Ingram and H. Cook [2] obtained a more general theorem by giving a characterization of all indecomposable continua by using coherent collections. An incorrect characterization, in terms of chains, of hereditarily indecomposable chainable continua was given in [1] and was corrected by A. Lelek in his review [MR **24**, #A2948] of that paper. Now a necessary and sufficient condition is presented for continua, not just those which are chainable, to be hereditarily indecomposable by referring to coherent collections of open sets in the statement of the condition.

All sets will be assumed to be subsets of compact metric spaces. Each compact continuum M is known to have a sequence $G_1, G_2, \cdots$ of finite open covers of M such that, for each natural number n, (1) the mesh of G_n is less than $1/n$, (2) each open set in G_n contains a point of M that is not contained by any other set in G_n, and (3) each open set in G_{n+1} has its closure contained as a subset of some element of G_n. A sequence of open covers with the above properties is called a defining sequence for M. Whenever T represents a collection of sets, the symbol T^* will represent the union of the elements of T. A collection T of sets is said to be coherent if and only if it is true that, whenever T is expressed as the union of two subcollections T_1 and T_2, then T_1^* and T_2^* have a point in common. A chain is a finite collection $\{c_1, c_2, \cdots, c_n\}$ of open sets such that, for $1 \leqq i < j \leqq n$, c_i intersects c_j if and only if $i = j - 1$. The elements of a chain are referred to as links. The definitions of all other terms used in this paper can be found in [3].

THEOREM. *A compact continuum M is hereditarily indecomposable if and only if there is a defining sequence $G_1, G_2, \cdots$ for M such that, for each*

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positive integer i, it is true that if A and B are elements of G_i with disjoint closures then there is an integer $j>i$ having the property that if L is a coherent collection of open sets of G_j with L^* intersecting both A and B then whenever L is expressed as the union of two coherent collections L_1 and L_2 there is an element C in G_i such that the elements of L_1 or L_2 together with C form a coherent collection the union of whose elements intersects both A and B.

PROOF. Suppose M is a compact continuum which has a defining sequence as described in the statement of the theorem. Assume that M has a subcontinuum M' which is the union of two proper subcontinua H and K. Choose a point x from $H-K$ and a point y from $K-H$. There is a positive integer i_0 such that each P and Q in G_{i_0} which contains x and y, respectively, have disjoint closures and each chain E of open sets of G_{i_0} with no more than three links has the property that if E^* contains x then E^* does not intersect K and if E^* contains y then E^* does not intersect H. Let A and B denote two elements of G_{i_0} which contain x and y, respectively. Choose any natural number $j_0>i_0$. The collection $L=\{g \in G_{j_0}:g$ intersects $M'\}$ is a coherent subcollection of G_{j_0} such that L^* intersects A and B. The union of the two coherent collections $L_1=\{g \in G_{j_0}:g$ intersects $H\}$ and $L_2=\{g \in G_{j_0}:g$ intersects $K\}$ is the collection L, and G_{i_0} does not contain an element C such that $L_1 \cup \{C\}$ or $L_2 \cup \{C\}$ is a coherent collection with elements intersecting A and B. This contradiction shows that M' must be indecomposable; therefore, M is hereditarily indecomposable.

Suppose M is a hereditarily indecomposable continuum and $G_1, G_2, \cdots$ is any defining sequence for M. Assume that, for some integer i_0, there are elements A and B of G_{i_0} which have disjoint closures and which have the property that, for each natural number k, a coherent subcollection of G_{i_0+k} is the union of two coherent collections L_{k1} and L_{k2} such that both A and B are intersected by $L^*_{k1} \cup L^*_{k2}$ and if C is an element of G_{i_0} then neither $L_{k1} \cup \{C\}$ nor $L_{k2} \cup \{C\}$ is a coherent collection which has the union of its elements intersecting both A and B. Notation is now chosen such that L^*_{k1} intersects A and L^*_{k2} intersects B for all natural numbers k. Due to L^*_{k1} intersecting A for each k, there is a point x_0 of $\bar{A}$ which is also a point of $\bigcap_{m=1}^{\infty} \mathrm{cl}(\bigcup_{k=m}^{\infty} L^*_{k1})$. A sequence of points $x_1, x_2, \cdots$ of M can be found such that the sequence converges to x_0 and such that there is a subsequence $k_1, k_2, \cdots$ of natural numbers with $x_i \in L^*_{k_i1}$ for each natural number i. The set $L^*_{k_i2}$ intersects B for each i; thus, there is a point y_0 of $\bar{B}$ and a subsequence $m_1, m_2, \cdots$ of $k_1, k_2, \cdots$ such that a sequence of points $y_1, y_2, \cdots$ of M converges to y_0 with $y_i \in L^*_{m_i2}$ for each i. Define M_1 to be the set $\bigcap_{j=1}^{\infty} \mathrm{cl}(\bigcup_{i=j}^{\infty} L^*_{m_i1})$ and M_2 to be the set $\bigcap_{j=1}^{\infty} \mathrm{cl}(\bigcup_{i=j}^{\infty} L^*_{m_i2})$. Notice that M_1 contains x_0 and M_2 contains y_0. Since each point which is a point of M_1 or M_2 is of distance zero from M, M_1 and M_2 are subsets of M. The point x_0 does not belong to M_2; otherwise, an element D of G_{i_0} that contains x_0 would intersect $L^*_{m_{i_1}2}$ for some i_1 which would imply that $L_{m_{i_1}2} \cup \{D\}$ would be a coherent collection with one of its elements intersecting A, namely D, and one of its elements intersecting B. Similarly, M_1 does not contain y_0. Since $L^*_{m_i1}$ and $L^*_{m_i2}$ intersect for each i, M_1 and M_2 have a common point. We now show that M_1 and M_2 are each connected. Suppose $M_1 = H \cup K$ where H and K are mutually exclusive closed sets. Suppose x_0 belongs to H. There are open sets D_H and D_K containing H and K as subsets, respectively, such that $\bar{D}_H$ does not intersect $\bar{D}_K$. Since D_H contains x_0, D_H contains all but at most finitely many of the terms of the sequence $x_1, x_2, \cdots$; thus, D_H intersects all but at most finitely many of the sets $L^*_{m_1 1}, L^*_{m_2 1}, \cdots$. Since D_K contains a point of M_1, D_K intersects infinitely many of the sets $L^*_{m_1 1}, L^*_{m_2 1}, \cdots$. This shows that there is a subsequence $n_1, n_2, \cdots$ of $m_1, m_2, \cdots$ such that each of D_H and D_K intersects $L^*_{n_i 1}$, for each i. There is an m_0 such that each chain of elements of G_m $(m \geqq m_0)$ has more than three links if one of its links intersects $\bar{D}_H$ and one of its links intersects $\bar{D}_K$. This shows that there is a sequence of open sets $g_1, g_2, \cdots$ such that $g_i \in L_{n_{m_0+i}1}$ and g_i does not intersect $\bar{D}_H$ or $\bar{D}_K$ for each i. Letting $z_i \in g_i$ for each i, there is a limit point of the set $\{z_1, z_2, \cdots\}$ which is a point of M_1 and which is not a point of D_H or D_K. This contradiction proves that M_1 is connected. The set M_2 is connected by a similar argument. Now $M_1 \cup M_2$ has been proven to be a decomposable subcontinuum of M. We must conclude that $G_1, G_2, \cdots$ has the property as described in the theorem.

References

1. Lida K. Barrett, *The structure of decomposable snakelike continua*, Duke Math. J. **28** (1961), 515–522. MR **24** #A2948.

2. W. T. Ingram and H. Cook, *A characterization of indecomposable compact continua*, Topology Conference (Arizona State Univ., Tempe, Ariz., 1967), Arizona State Univ., Tempe, Ariz., 1968, pp. 168–169. MR **40** #6513.

3. R. L. Moore, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 13, Amer. Math. Soc., Providence, R.I., 1962. MR **27** #709.

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