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WILD CELLS IN E^4 IN WHICH EVERY ARC IS TAME

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ABSTRACT. Seebeck has proved that if an m-cell C in Euclidean n-space E^n factors k-times, $m \leqq n-2$, and $n \geqq 5$, then every embedding of a compact k-dimensional polyhedron in C is tame relative to E^n. We prove the analogous result for $n=4$ and $m \leqq 3$.

Following Seebeck we say that an m-cell C in E^n *factors k times* if for some homeomorphism h of E^n onto itself and some $(m-k)$-cell B in E^{n-k}, $h(C)=B \times I^k$, where I^k denotes the k-fold product of the interval I naturally embedded in E^k and where

$$B \times I^k \subset E^{n-k} \times E^k = E^n$$

is the product embedding.

A metric space X is said to be *uniformly simply connected*, denoted 1-ULC, if to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that every map f of the boundary of a 2-cell D into a δ-subset of X can be extended to a map F of D into an ε-subset of X. Furthermore, a subset A of X is said to be *uniformly simply connected in X*, written 1-ULC in X, if to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that every map f of the boundary of D into a δ-subset of A can be extended to a map F of D into an ε-subset of X.

The reader is referred to [**5**] for definitions of other terms used here.

The techniques found in this paper are related to those used by Sher in [**6**], in which he established the analogue of Theorem 3 of [**5**] for the case $n=4$.

THEOREM. *If C is an m-cell in E^4 $(m \leqq 3)$ that factors* 1 *time, then every embedding of a compact* 1*-complex in C is tame relative to E^4.*

PROOF. Let B denote an $(m-1)$-cell in E^3 and P an arc in $B \times I \subset E^3 \times E^1 = E^4$. It suffices to prove that P is tame in the two cases $m=2$ and $m=3$.

In case $m=2$ we apply techniques from the proof of Lemma 2 of [**5**], as were required in proving Theorem 5 of [**5**], to obtain a 0-dimensional F_σ-subset F of B such that $(E^3-B) \cup F$ is 1-ULC. In case $m=3$ we can

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show that B contains a 0-dimensional F_σ-set F such that E^3-B is 1-ULC in $(E^3-B)\cup F$. One way to do this is to split E^3 along B to obtain a 3-cell K that maps nicely onto B under a map f, as stated in Theorem 9 of [3], to apply Theorem 4.2 of [1] to obtain a 0-dimensional F_σ set F' in the boundary of K such that $(E^3-K)\cup F'$ is 1-ULC, to notice that $f(F')$ is again a 0-dimensional F_σ-set, and to deduce from the properties of the map f that E^3-B is 1-ULC in $(E^3-B)\cup f(F')$. In either case we express F as the union of compact sets $F_1, F_2, \cdots$.

Define a subset T of P as the set of all points x in P for which there exist a neighborhood U of x relative to P and a point y in E^3 such that $U\subset\{y\}\times I$. Obviously T is an open subset of P. Let π denote the projection of $E^4=E^3\times E^1$ onto E^3, and let $\{s_j\}_{j=1}^{\infty}$ denote a countable dense subset of Int I.

We interpose two lemmas and provide a crude indication of the proofs.

LEMMA 1. *Let i and j denote positive integers, ε a positive number, and g a homeomorphism of E^4 onto itself such that $g(B\times I)=B\times I$ and $\pi g=\pi$. There exists an ε-push f of $(E^4, F_i\times\{s_j\})$ such that $f(B\times I)=B\times I$, $\pi f=\pi$, and $fg(P-T)\cap(F_i\times\{s_j\})=\varnothing$.*

PROOF. The set F_j can be covered by a finite collection of pairwise disjoint, small, open (relative to E^3) sets $B_1, \cdots, B_t$ such that for $k=1, \cdots, t$ there exists a positive number $\delta_k<\varepsilon$ for which

$$(B_k\times\{s_j-\delta_k\})\cap g(P-T)=\varnothing.$$

By isotoping $(F_i\cap B_k)\times\{s_j-\delta_k\}$ to $(F_i\cap B_k)\times\{s_j\}$ while moving no point of $(E^3-\bigcup B_k)\times E^1$, we can construct an ε-push f of $(E^4, F_i\times\{s_j\})$ with the required properties.

LEMMA 2. *There exists a homeomorphism h of $E^3\times E^1$ onto itself such that* (1) $h(B\times I)=B\times I$, (2) $\pi h=\pi$, (3) $h(P-T)\cap(F\times\{s_j\})=\varnothing$, *and* (4) $h(P)\cap(B\times\{s_j\})$ *is* 0-*dimensional* $(j=1, 2, \cdots)$.

PROOF. We apply Lemma 1 repeatedly to construct h as the end of an isotopy. Let $\varepsilon>0$. First we push P off $F_1\times\{s_1\}$ with an $\varepsilon/2$-push f_1. Then we push $f_1(P)$ off $F_2\times\{s_1\}$ with a push f_2 no more than $\varepsilon/4$ but so small that $f_2f_1(P)$ cannot intersect $F_1\times\{s_1\}$. We push $f_2f_1(P)$ off $F_3\times\{s_1\}$ with a very small push f_3, and we continue in this fashion, taking sufficient care so that the limit of $f_n\circ\cdots\circ f_1$ as $n\to\infty$ is an ε-push h, of $(E^3, F\times\{s_1\})$ such that (1) $h_1(B\times I)=B\times I$, (2) $\pi h_1=\pi$, and (3) $h_1(P-T)\cap(F\times\{s_1\})=\varnothing$. If $h_1(P)\cap(B\times\{s_1\})$ fails to be 0-dimensional, we push slightly the interiors of the intervals of intersection to overcome this, while preserving the other three properties.

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The remaining pushes will not change the map on $h_1^{-1}(E^3\times\{s_1\})$. Using the techniques of the preceding paragraph, we define a $(3\varepsilon/2)$-push h_2 of $(E^4, F\times\{s_2\})$ such that (1) $h_2(B\times I)=B\times I$, (2) $\pi h_2=\pi$, (3) $h_2(P-T)\cap(F\times\{s_2\})=\varnothing$, (4) $h_2(P)\cap(B\times\{s_2\})$ is 0-dimensional, and

$$h_2^{-1}(E^3 \times \{s_1\}) = h_1^{-1}(E^3 \times \{s_1\}). \tag{5}$$

The pattern should be clear. The remaining pushes will not change the map on $h_2^{-1}(E^3\times\{s_1, s_2\})$. We obtain a $(7\varepsilon/4)$ push h_3 that improves the intersections at the s_3-level. In continuing this process, we exercise controls on the epsilonics to guarantee that the limiting final state of this sequence of pushes is the required homeomorphism.

Returning to the proof of the theorem, we let h be a homeomorphism satisfying the conclusions of Lemma 2. In the case $m=2$ it is easy to show that $(E^3\times\{s_i\})-h(P-T)$ is 1-ULC for each i; similarly, in the case $m=3$ it is easy to show that $(E^3-B)\times\{s_i\}$ is 1-ULC in $(E^3\times\{s_i\})-h(P-T)$, and by [**3**], $(E^3\times\{s_i\})-h(P-T)$ is 1-ULC. In either case, consequently, for a map f of a 2-cell D into $(E^3\times\{s_i\}-h(P-T))$, each point q of $f(D)\cap h(P)$ is an isolated point of $h(T)$, because each component of $h(T)$ is projected to a singleton by π. In addition, since some neighborhood of q $h(P)$ intersects $E^3\times\{s_i\}$ only at q, it is easy to adjust the map f slightly so that

$$f(D) \subset (E^3 \times \{s_i\}) - h(P).$$

Thus, $(E^3\times\{s_i\})-h(P)$ is 1-ULC $(i=1, 2, \cdots)$. It follows from a recent result of Bryant [**2**, Corollary 2] (a convenient restatement of this is given as Theorem 1 of [**6**]) that $h(P)$ is tame. Clearly, P must be tame also.

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