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DERIVATION MODULES OF FREE JOINS AND $\mathfrak{m}$-ADIC COMPLETIONS OF ALGEBRAS

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ABSTRACT. A free commutative join of subalgebras corresponds to a direct sum of submodules in a universal derivation module. In particular, indeterminates of a polynomial ring correspond to elements of a linearly independent set in a universal derivation module. As an application, a simple proof of the uniqueness of cardinalities of indeterminates of a polynomial ring can be obtained by using that of linear bases of a free module over a commutative ring. Similar observations are made for $\mathfrak{m}$-adic completions of algebras and their derivation modules. Also, the module of linear differential forms of an $\mathfrak{m}$-adic completion of an algebra is studied.

1. **Preliminaries.** Let R be a commutative ring with unit, and A a unitary commutative R-algebra. By a *derivation module* of A, we mean a pair (M, d) where M is an A-module and $d:A\to M$ an R-derivation. A *derivation module homomorphism* $f:(M, d)\to(N, \partial)$ is an A-homomorphism $f:M\to N$ such that $f\circ d=\partial$. In the category of derivation modules of A, there exists a universal object [**5**, p. 101] which is uniquely determined by A up to unique derivation module isomorphisms, and this is called a *universal derivation module of* A.

We notice that a set X of indeterminates of a polynomial ring $R[X]$ corresponds to a linear basis in a universal derivation module of $R[X]$; and that we can prove the uniqueness of the cardinalities of sets of indeterminates by using that of bases of a module. In fact, let U be a free $R[X]$-module with a set $\{u_x:x \in X\}$, where $u_x=u_{x'}$ iff $x=x'$ for x, $x' \in X$, as a basis and $d:R[X]\to U$ be an R-derivation defined by $df=\sum_x (\partial f/\partial x)u_x$, $f\in R[X]$, then (U, d) is a universal derivation module of $R[X]$ [**1**, p. 8, Satz 2]. Hence the cardinality of X is equal to the dimension of U over $R[X]$. For any two sets X and Y, let $R[X]\cong R[Y]$ as R-algebras (R is not necessarily an integral domain). Suppose that (U, d) and (V, δ) are universal derivation modules of $R[X]$ and $R[Y]$ respectively, then V can be

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made into an $R[X]$-module by an isomorphism of $R[X]$ onto $R[Y]$; and $R[X] \cong R[Y]$ also implies that $U \cong V$ as $R[X]$-modules by the uniqueness of universal derivation modules. It follows that $\dim U = \dim V$, and hence Card $X = \dim U = \dim V =$ Card Y.

The following questions arise out of this observation: (1) What in an algebra corresponds to a direct sum in a universal derivation module? (§2). (2) Can we have a similar argument concerning the cardinalities of indeterminates in case of rings of formal power series? (§3). For rings of formal power series, instead of a universal derivation module we use a universal Hausdorff derivation module which is a universal object in a subcategory of the category of derivation modules. In fact, it is known in [**1**, p. 16] that the cardinality of a basis of a universal derivation module of a ring $k[[x]]$ of formal power series in one indeterminate over a field k of characteristic zero is infinite. From this we see that a universal derivation module is too large to be used for the cardinality argument in case of formal power series. We find that the subcategory consisting of Hausdorff derivation modules serves well in this case. (3) It is well known that if a universal derivation module (U, d) of A is finitely generated and projective, then the A-module $\mathscr{D}_A^*$ of linear differential forms of A is isomorphic to U. Is there a similar property to this for universal Hausdorff derivation modules? (§4).

Conventions. Throughout this paper, R will denote a commutative ring with unit; and all algebras, subalgebras and algebra homomorphisms are assumed to be unitary.

2. **Free commutative join of algebras.** In the category of commutative R-algebras, if an R-algebra A is a coproduct of a family $(A_\alpha)_{\alpha \in I}$ of its subalgebras, then A is called a *free commutative join* of $(A_\alpha)_{\alpha \in I}$ [**5**, p. 93]. For example, a polynomial algebra $R[X]$ with X as a set of indeterminates is a free commutative join of a family $(R[x])_{x \in X}$ of subalgebras. Also, if an R-module M is a direct sum of a family $(M_\alpha)_{\alpha \in I}$ of its submodules, a symmetric algebra $S(M)$ is a free commutative join of a family $(S(M_\alpha))_{\alpha \in I}$ of its subalgebras.

Let $(A_\alpha)_{\alpha \in I}$ be a family of commutative R-algebras; $\bigotimes_{\alpha \in I} A_\alpha$ a tensor product of $(A_\alpha)_{\alpha \in I}$, i.e. in the R-module $\bigotimes_{\alpha \in I} A_\alpha$, a multiplication is defined by $(\otimes a_\alpha)(\otimes b_\alpha) = \otimes a_\alpha b_\alpha$ for $\otimes a_\alpha$, $\otimes b_\alpha \in \bigotimes_{\alpha \in I} A_\alpha$. Obviously it is a commutative algebra.

THEOREM 1. *If a commutative R-algebra A is a free commutative join of a family $(A_\alpha)_{\alpha \in I}$ of its subalgebras, then A is isomorphic to a subalgebra A' of $\bigotimes_{\alpha \in I} A_\alpha$ generated by all elements $\otimes a_\alpha \in \bigotimes_{\alpha \in I} A_\alpha$ such that $a_\alpha = 1$ for all but finitely many $\alpha \in I$.*

PROOF. Let C be any commutative R-algebra, and $(f_\alpha)_{\alpha\in I}$ a family of algebra homomorphisms $f_\alpha: A_\alpha \to C$. Let $g_0: \prod_{\alpha\in I} A_\alpha \to C$ be a multilinear mapping such that $g_0((a_\alpha)_{\alpha\in I}) = \prod_{\alpha\in I} f_\alpha(a_\alpha)$ if $a_\alpha = 1$ for all but finitely many $\alpha \in I$ and $g_0((a_\alpha)_{\alpha\in I}) = 0$ otherwise; and let $g: \bigotimes_{\alpha\in I} A_\alpha \to C$ be the R-linearization of g_0. Then g is an algebra homomorphism such that $g \circ j_\alpha = f_\alpha$ for each $\alpha \in I$, where $j_\beta: A_\beta \to \bigotimes_{\alpha\in I} A_\alpha$ is the canonical homomorphism defined by $j_\beta(a) = \otimes\, a_\alpha$ where $a_\beta = a$, $a_\alpha = 1$ for all $\alpha \neq \beta$. Let $f = g|A'$ (restriction). Then $f: A' \to C$ is an R-algebra homomorphism such that $f \circ j_\alpha = f_\alpha$ for every $\alpha \in I$. The uniqueness of such f is obvious from the fact that A' is generated by all $j_\alpha(A_\alpha)$, $\alpha \in I$. We will prove that each j_α is one-to-one. We consider a particular case where $C = A$ and $f_\alpha: A_\alpha \to A$ is the natural injection for each $\alpha \in I$, then $f: A' \to A$ is an algebra homomorphism such that $f \circ j_\alpha = f_\alpha$. Since f_α is one-to-one, j_α is one-to-one. It follows that A' is also a free commutative join of $(A_\alpha)_{\alpha\in I}$ and hence $A \cong A'$.

Q.E.D.

Convention. From now on, if A is a free commutative join of $(A_\alpha)_{\alpha\in I}$, then we will assume that A is imbedded into $\bigotimes_{\alpha\in I} A_\alpha$. So $A \subseteq \bigotimes_{\alpha\in I} A_\alpha$.

LEMMA 1. *Let a be a free commutative join of $(A_\alpha)_{\alpha\in I}$. Suppose that B is an R-submodule of $\bigotimes_{\alpha\in I} A_\alpha$ generated by all elements $\otimes\, b_\alpha \in \bigotimes_{\alpha\in I} A_\alpha$ such that $b_\alpha \neq 1$ for infinitely many $\alpha \in I$. Then*

$$\bigotimes_{\alpha\in I} A_\alpha = A + B \quad (\textit{direct}).$$

PROOF. It is evident that $\bigotimes_{\alpha\in I} A_\alpha = A + B$. To prove the directness of the sum, let $x \in A \cap B$. Then $x \in A$ implies that x can be expressed as a sum of elements of the form $\otimes\, a_\alpha$ where $a_\alpha = 1$ for all but finitely many $\alpha \in I$; $x \in B$ implies that x can be expressed as a sum of the form $\otimes\, b_\alpha$ where $b_\alpha \neq 1$ for infinitely many $\alpha \in I$. Let $x = \sum \otimes\, a_\alpha = \sum \otimes\, b_\alpha$. Then $\sum \otimes\, a_\alpha - \sum \otimes\, b_\alpha = 0$. In the free module F with $\prod_{\alpha\in I} A_\alpha$ as basis, of which $\bigotimes_{\alpha\in I} A_\alpha$ is the quotient by its usual construction,

$$(2.1) \quad \sum (a_\alpha)_{\alpha\in I} - \sum (b_\alpha)_{\alpha\in I} = \sum \pm((x_\alpha)_{\alpha\in I} - \lambda(y_\alpha)_{\alpha\in I} - \mu(z_\alpha)_{\alpha\in I}),$$

where $x_\beta = \lambda y_\beta + \mu z_\beta$ for one $\beta \in I$ and $x_\alpha = y_\alpha = z_\alpha$ for all $\alpha \neq \beta$. Let $h: F \to F$ be an R-linear mapping such that for any $(u_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} A_\alpha$,

$$\begin{aligned} h((u_\alpha)_{\alpha\in I}) &= (u_\alpha)_{\alpha\in I} \quad \text{if } u_\alpha = 1 \text{ for all but finitely many } \alpha \in I, \\ &= 0 \qquad\qquad \text{if } u_\alpha \neq 1 \text{ for infinitely many } \alpha \in I. \end{aligned}$$

Act h on both the sides of the equation (2.1). Then on the right-hand side of (2.1), every term $((x_\alpha)_{\alpha\in I} - \lambda(y_\alpha)_{\alpha\in I} - \mu(z_\alpha)_{\alpha\in I})$ vanishes, except those terms for which $x_\alpha = 1$ for all but finitely many $\alpha \in I$. Hence

$$\sum (a_\alpha)_{\alpha\in I} = \sum \pm((x_\alpha)_{\alpha\in I} - \lambda(y_\alpha)_{\alpha\in I} - \mu(z_\alpha)_{\alpha\in I}),$$

the sum of all terms where $x_\alpha=1$ for all but finitely many $\alpha \in I$. It follows from this that $x=\sum \otimes a_\alpha=0$ in $\otimes_{\alpha\in I} A_\alpha$. Hence the sum is direct. Q.E.D.

LEMMA 2. *Let S be any subalgebra of A, M an S-module; then the canonical S-homomorphism $A\otimes_S M\to(\otimes_{\alpha\in I} A_\alpha)\otimes_S M$, $a\otimes x\mapsto a\otimes x$ is one-to-one.*

PROOF. From Lemma 1, $\otimes_{\alpha\in I} A_\alpha=A+B$ (direct). Let $\pi:\otimes_{\alpha\in I} A_\alpha\to A$ be the first projection. Let $\sum a\otimes x \in A\otimes_S M$; and suppose that the image $\sum a\otimes x$ (in $(\otimes_{\alpha\in I} A_\alpha)\otimes_S M$) under the canonical homomorphism is zero. Then in the free module F with $(\otimes_{\alpha\in I} A_\alpha)\times M$ as basis, of which $(\otimes_{\alpha\in I} A_\alpha)\otimes_S M$ is the quotient by its usual construction,

$$\begin{aligned} \sum (a, x) = {} & \sum \pm((b+c, y)-(b,y)-(c,y)) \\ & + \sum \pm((d, z+u)-(d,z)-(d,u)) \\ & + \sum \pm((es, v)-(e, sv)) \end{aligned} \tag{2.2}$$

where $b, c, d, e \in \otimes_{\alpha\in I} A_\alpha$, $s\in S$, and $y, z, u, v \in M$.

Let $h:F\to F$ be the S-homomorphism such that $h((f, w))=(\pi(f), w)$ for all $(f, w)\in(\otimes_{\alpha\in I} A_\alpha)\otimes_S M$. By acting h on both the sides of the equation (2.2), we can easily see that $\sum a\otimes x=0$ in $A\otimes_S M$. This implies that the canonical homomorphism is one-to-one.

Convention. We assume that $A\otimes_S M\subseteq(\otimes_{\alpha\in I} A_\alpha)\otimes_S M$.

THEOREM 2. *Let A be a free commutative join of a family $(A_\alpha)_{\alpha\in I}$ of its subalgebras, and (U_α, d_α) a universal derivation module of A_α for each $\alpha\in I$. Let*

(1) *$U=\bigoplus_{\alpha\in I}(A\otimes_{A_\alpha} U_\alpha)$, and*

(2) *$d:A\to U$ is an R-derivation defined by*

$$d\left(\sum_\alpha a_{\alpha_1}\cdots a_{\alpha_{n_\alpha}}\right)=\sum_\alpha\left(\sum_{j=1}^{n} a_{\alpha_1}\cdots \hat{a}_{\alpha_j}\cdots a_{\alpha_{n_\alpha}}\otimes d_{\alpha_j}a_{\alpha_j}\right)$$

where $a_{\alpha_i}\in A_{\alpha_i}$, $\alpha_i\in I$, and $\hat{a}_{\alpha_j}$ denotes the omission of a_{α_j}. Then (U, d) is a universal derivation module of A.

PROOF. We first show that d is well defined. Let $f_0:\prod_{\alpha\in I} A_\alpha\to(\otimes_{\alpha\in I} A_\alpha)\otimes_{A_\beta} U_\beta$ be an R-multilinear mapping defined by

$$f_0((a_\alpha)_{\alpha\in I})=\left(\prod_{\alpha\in I;\alpha\neq\beta} j_\alpha(a_\alpha)\right)\otimes d_\beta a_\beta,$$

where $j_\alpha:A_\alpha\to\otimes_{\alpha\in I} A_\alpha$ is the canonical homomorphism as in the proof of Theorem 1. Let f be the linearization of f_0; and let $D_\beta:A\to A\otimes_{A_\beta} U_\beta$ be a mapping such that $D_\beta=f|A$ for each $\beta\in I$. We define $d:A\to\bigoplus_{\alpha\in I}(A\otimes_{A_\alpha} U_\alpha)$

by

$$d(a) = \sum_{\alpha\in I} D_\alpha(a), \qquad a \in A.$$

Then by a straightforward calculation, d is the R-derivation in (2). Hence d is well defined.

To show that (U, d) is universal, let (M, ∂) be an arbitrary derivation module of A. Since each (U_α, d_α) is universal, there exists an A_α-homomorphism $f_\alpha: U_\alpha \to M$ such that $f_\alpha \circ d = \partial | A_\alpha$ for each $\alpha \in I$. Let $g_\alpha: A \otimes_{A_\alpha} U_\alpha \to M$ be an A-homomorphism such that $g_\alpha(a\otimes x) = af_\alpha(x)$, $a \in A$, $x \in U_\alpha$. We define $g: \bigoplus_{\alpha\in I} (A\otimes_{A_\alpha} U_\alpha) \to M$ by

$$g\left(\sum_\alpha u_\alpha\right) = \sum g_\alpha(u_\alpha), \qquad u_\alpha \in A\otimes_{A_\alpha} U_\alpha.$$

Then $g: U \to M$ is a unique A-homomorphism such that $g\circ d = \partial$. Q.E.D.

3. **$\mathfrak{m}$-adic completions.** Let A be a commutative R-algebra with $\mathfrak{m}$-topology where $\mathfrak{m}$ is a proper ideal of A. We assume that $\bigcap_{n=0}^\infty \mathfrak{m}^n = 0$ so that the $\mathfrak{m}$-topology is Hausdorff. A derivation module (M, d) of A is said to be *Hausdorff* if M is Hausdorff with respect to $\mathfrak{m}$-topology, i.e. $\bigcap_{n=0}^\infty \mathfrak{m}^n M = 0$. We will confine ourselves to the category of Hausdorff derivation modules. Then in this category, a universal object which is called a universal Hausdorff derivation module exists and is uniquely determined by A up to unique derivation module isomorphisms. The proof of Theorems 3 and 4 are similar to those for Hausdorff complexes in [**7**, pp. 185–194].

THEOREM 3. *Let (U, d) be a universal derivation module of A. Then (V, δ) is a universal Hausdorff derivation module of A if $V = U/\bigcap_{n=0}^\infty \mathfrak{m}^n U$ and $\delta = \nu\circ d$, where $\nu: U \to V$ is the natural homomorphism.*

PROOF. Obvious.

For a Hausdorff derivation module (H, ∂) of A, ∂ is uniformly continuous. In fact, $\partial\mathfrak{m}^n \subseteq \mathfrak{m}^{n-1}H$ and hence if we denote the order functions on both A and H by v (cf. [**8**, p. 249]), for any $x, y \in A$, $v(\partial x - \partial y) = v(x-y) - 1$. Let e be a real number such that $e > 1$, and define a distance in A by $D(x, y) = e^{-v(x-y)}$, $x, y \in A$; in H by $D(u, v) = e^{-v(u-v)}$, $u, v \in H$. For any $\varepsilon > 0$, put $\eta = \varepsilon/e$. Then for any $x, y \in A$ with $D(x, y) < \eta$, $D(\partial x - \partial y) < \varepsilon$; ∂ is uniformly continuous. Let $\hat{A}$ and $\hat{H}$ be $\mathfrak{m}$-adic completions. $\hat{H}$ can be made into an $\hat{A}$-module and let $\hat{\partial}$ be the continuous extension of ∂ to $\hat{A}$. Then $\hat{\partial}\hat{x} = \lim_{n\to\infty} \partial x_n$, where $\hat{x} = \lim_{n\to\infty} x_n$, $x_n \in A$; and it is easy to see that $\hat{\partial}$ is an R-derivation. Hence $(\hat{H}, \hat{\partial})$ is a derivation module of $\hat{A}$. In general, $(\hat{H}, \hat{\partial})$ is not a Hausdorff derivation module with respect to

$(\hat{A}\mathfrak{m})$-topology. However, if $\mathfrak{m}$ and H are finitely generated it is Hausdorff as is explained in [**8**, p. 257]; the topology of $\hat{H}$ considered as the metric completion of H is equal to the $(\hat{A}\mathfrak{m})$-topology of $\hat{H}$, and hence is Hausdorff.

THEOREM 4. *Let A be a commutative R- algebra and $\mathfrak{m}$ a proper ideal of A. If A is a Noetherian ring such that a universal Hausdorff derivation module (V, δ) of A with respect to $\mathfrak{m}$-topology is finitely generated, then $(\hat{V}, \hat{\delta})$ is a universal Hausdorff derivation module of $\hat{A}$ with respect to $(\hat{A}\mathfrak{m})$-topology.*

PROOF. Let (N, ∂) be an arbitrary Hausdorff derivation module of $\hat{A}$ with respect to $(\hat{A}\mathfrak{m})$-topology. Then the restriction $\partial_0: A \to N$ of ∂ to A is an R-derivation of A. Since (V, δ) is a universal Hausdorff derivation module of A, there exists a unique A-homomorphism $f: V \to N$ such that $f \circ \delta = \partial_0$. We define $\varphi: \hat{A} \times V \to N$ by $\varphi(\hat{a}, v) = \hat{a}f(v)$, $\hat{a} \in A$, $v \in V$; then φ is a bilinear mapping. Let $\hat{f}: \hat{A} \otimes_A V \to N$ be the linearization of φ; then $\hat{f}$ is automatically an $\hat{A}$-homomorphism. Since A is a Noetherian ring and V is finitely generated, it is known in [**3**, p. 68, Théorème 3] that $\hat{V} \cong \hat{A} \otimes_A V$. Hence $\hat{f}$ can be considered as an $\hat{A}$-homomorphism from $\hat{A}$ into N. $\hat{f}$ and $\hat{\delta}$ are continuous with respect to $(\hat{A}\mathfrak{m})$-topology and hence it is clear that $\hat{f} \circ \hat{\delta} = \partial$. The uniqueness of such $\hat{f}$ follows from the fact that $\hat{V}$ is generated by $\hat{\delta}\hat{A}$ as $\hat{A}$-module. Q.E.D.

COROLLARY 1. *Let R be a commutative Noetherian ring with unit, and (V, δ) be a universal Hausdorff derivation module of a ring $R[[x_1, \cdots, x_n]]$ of formal power series in n indeterminates. Then V is isomorphic to a free $R[[x_1, \cdots, x_n]]$-module with a set $\{v_1, \cdots, v_n\}$ as a basis and $\delta: R[[x_1, \cdots, x_n]] \to V$ is an R-derivation defined by $\delta f = \sum_{i=1}^{n} (\partial f / \partial x_i) v_i$, $f \in R[[x_1, \cdots, x_n]]$ where $\partial f / \partial x_i$ is a partial derivative* [**2**, p. 62].

PROOF. $R[[x_1, \cdots, x_n]]$ is the $(x_1, \cdots, x_n)$-adic completion of $R[x_1, \cdots, x_n]$, and hence it is an immediate consequence of Theorem 4.

COROLLARY 2. *Let R be a commutative Noetherian ring with unit. Then $R[[x_1, \cdots, x_n]] \cong R[[y_1, \cdots, y_m]]$ as R-algebras imply that $n = m$.*

PROOF. Similar to the proof for polynomial rings in §1.

4. **Linear differential forms.** Let $\mathscr{D}_A$ be the A-module of all R-derivations $D: A \to A$. The A-module $\mathscr{D}_A^* = \operatorname{Hom}_A(\mathscr{D}_A, A)$ is called the module of linear differential forms on A. In this section, we give a partial answer to the question: When is $\mathscr{D}_A^*$ isomorphic to a universal Hausdorff derivation module of A?

A mapping $d: A \to \mathscr{D}_A^*$ defined by $d(a)(D) = D(a)$, $a \in A$, $D \in \mathscr{D}_A$, is an R-derivation, and hence $(\mathscr{D}_A^*, d)$ is a derivation module of A.

THEOREM 5. *Let A be a commutative R-algebra such that it is a Hausdorff space with respect to* $\mathfrak{m}$*-topology. If a universal Hausdorff derivation module* (V, δ) *of A is finitely generated and projective, then*

(1) $\mathscr{D}_A^*$ *is a Hausdorff space with respect to* $\mathfrak{m}$*-topology,*

(2) $(V, \delta) \cong (\mathscr{D}_A^*, d)$.

PROOF. (1) For any derivation $D: A \to A$, (A, D) is a Hausdorff derivation module. Hence there exists a unique A-homomorphism $h_D \in V^*$ such that $h_D \circ \delta = D$. We define a mapping $\varphi: \mathscr{D}_A \to V^*$ by $\varphi(D) = h_D$, then obviously φ is an A-homomorphism. Moreover, φ is an isomorphism; one-to-one, since $h_D = 0$ implies $D = h_D \circ \delta = 0$; onto, since for any $h \in V^*$, $h \circ \delta \in \mathscr{D}_A$ and $\varphi(h \circ \delta) = h$. Hence $\mathscr{D}_A \cong V^*$ and from this $\mathscr{D}_A^* \cong V^{**} \cong V$ (V is finitely generated and projective and hence $V \cong V^{**}$). V is Hausdorff by assumption, and hence so is $\mathscr{D}_A^*$.

(2) $(\mathscr{D}_A^*, d)$ is a Hausdorff derivation module and (V, δ) is universal. Hence there exists an A-homomorphism $f: V \to \mathscr{D}_A^*$ such that $f \circ \delta = d$. It is sufficient to show that f is an isomorphism. Let $g: V \to V^{**}$ be the natural homomorphism defined by $g(v) = v^{**}$, $v \in V$, where $v^{**}(h) = h(v)$, $h \in V^*$; g is an isomorphism, since V is finitely generated and projective. Also, let $\psi^*: \mathscr{D}_A^* \to V^{**}$ be the dual homomorphism of $\psi: V^* \to \mathscr{D}_A$ defined by $\psi(h) = h \circ \delta$ (in fact, $\psi = \varphi^{-1}$, φ in (1)). Then $\psi^* \circ f = g$. For, $\psi^* \circ f(\delta a) = \psi^*(da) = (da) \circ \psi$, $a \in A$; and $((da) \circ \psi)(h) = (da)(h \circ \delta) = (h \circ \delta)(a) = (\delta a)^{**}(h) = (g(\delta a))(h)$ for all $h \in V^*$, and from this $(da) \circ \psi = g(\delta a)$ and $\psi^* f(\delta a) = g(\delta a)$. Hence $\psi^* \circ f$ coincides with g on δA, but V is generated by δA as A-module and hence $\psi^* \circ f = g$. Since g and ψ^* are isomorphisms, f is an isomorphism. Q.E.D.

COROLLARY. *Let A be a commutative Noetherian R-algebra such that a universal Hausdorff derivation module (V, δ) of A is finitely generated and projective, then the $\hat{A}$-module of linear differential forms of $\hat{A}$ is isomorphic to $(\hat{V}, \hat{\delta})$, and hence $\mathscr{D}_{\hat{A}}^* \cong \hat{V}$.*

PROOF. Immediate consequence of Theorems 4 and 5.

COROLLARY. *Let R be a commutative Noetherian ring. Then the module of linear differential forms of a ring $R[[x_1, \cdots, x_n]]$ of formal power series is isomorphic to a free $R[[x_1, \cdots, x_n]]$-module of dimension n (this is a well-known fact in* [**2**, p. 62]).

PROOF. Obvious from the previous corollary.

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