ON FINITE INVARIANT MEASURES FOR SETS OF MARKOV OPERATORS

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Abstract. A. Brunel [1] proved that a Markovian operator P has an invariant measure if and only if each convex combination of iterates \( \sum_{n=0}^{\infty} \alpha_n P^n \) is conservative. In the present paper this result is generalized for any commutative semigroup of Markovian operators:

Let \( \Pi \) be a semigroup; there exists a common invariant measure for \( \Pi \) if and only if each convex combination \( \sum_{n=1}^{\infty} \alpha_n P_n \), where \( \{P_n\} \subset \Pi \), is conservative.

1. Definitions and notations. Let \((X, \Sigma, m)\) be a finite measure space. A Markov operator \( P \) is a positive contraction on \( L_1(X, \Sigma, m) \) i.e., (i) \( \|P\| = 1 \), (ii) \( u \geq 0 \Rightarrow uP \geq 0 \). We shall use the notations of [3], so the operator adjoint to \( P \) which is defined in \( L_\infty(m) \) will also be denoted by \( P \) to the left side of the variable. Thus \( \langle uP, f \rangle = \langle u, Pf \rangle \), \( u \in L_1(m), f \in L_\infty(m) \).

Let \( \Sigma_i(P) = \{ A \in \Sigma \mid P1_A = 1_A \text{ a.e.} \} \). If \( P \) is conservative, then \( \Sigma_i(P) \) is a field.

Let us consider the commutative semigroup \( \Pi \) of conservative Markov operators. The invariant sets of \( \Pi \) is the collection \( \Sigma_i = \cap_{P \in \Pi} \Sigma_i(P) \).

Let \( \Pi \) be the convex hull of \( \Pi \): (i) \( \Pi \subset \Pi \), (ii) \( \{P_n\} \subset \Pi \) and \( P_n \rightarrow P \) in operator norm then \( P \in \Pi \). (iii) \( P_1, P_2 \in \Pi, 0 < \alpha, \beta < 1 \) and \( \alpha + \beta = 1 \) then \( \alpha P_1 + \beta P_2 \in \Pi \). (iv) \( \Pi \) is minimal under those conditions.

A measure \( \mu \) is said to be invariant for \( \Pi \) if \( \mu P = \mu \) for every \( P \in \Pi \). Clearly, \( \Pi \) is a semigroup and if \( \mu \) is a finite invariant measure for \( \Pi \) it is invariant for \( \Pi \).

In this paper we prove that if there is no finite invariant measure for \( \Pi \) then there exists an operator \( Q \in \Pi \) such that the dissipative part of \( Q, D = X \). (For definitions see [3].)

Remark. In [1] it is proved for \( \Pi = \{P^n\} \), where \( P \) is a given Markov operator, that if it has no finite invariant measures, then there exists \( \Pi \subset \Pi \), such that the dissipative part of \( Q, D \neq \emptyset \).

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2. Conservative operators and invariant measures.

**Lemma 2.1.** If there are no finite invariant measures for \( \Pi \), then there exists \( \{P_n\} \subset \Pi \) and \( 0 \leq f \in L_\infty(m) \) such that \( \sum_{n=1}^{\infty} P_n f \in L_\infty(m) \).

**Proof.** In [4] it is proved that if there exists no finite invariant measure then there exists \( 0 \leq g \in L_\infty(m) \) such that \( \inf \{ \int P g \, dm \mid P \in \Pi \} = 0 \). Hence by slight modifications of the proof of Lemma C, Chapter IV of [3], or of the more elegant proof of this lemma which appears in [2], Lemma 2.1 can be proved.

Let us define the space:

\[
L = \text{span}\{(I - P)L_\infty(m) \mid P \in \Pi\}.
\]

The orthogonal complement of \( L \) is

\[
L^\perp = \{ v \in L_\infty^*(m) \mid vP = v \forall P \in \Pi \}.
\]

\( L^\perp \) is the space of the charges (finitely additive measures). If \( P1 = 1 \), as it is in the conservative case, then \( vP = v \) if and only if \( v^+P = v^+ \). Define:

\[
M = \{ v \in L_\infty^*(m) \mid v \geq 0, \| v \| = 1, vP = v \forall P \in \Pi \}.
\]

It is easy to show that:

\[
f \in L \iff \langle v, f \rangle = 0, \quad v \in M.
\]

**Lemma 2.2.** If there exists no finite invariant measures for \( \Pi \), then there exists \( 0 \leq f \geq 0 \) such that \( f \in L \).

**Proof.** Let \( f \) be the function of Lemma 2.1. It is clear that \( \langle v, f \rangle = 0 \) for each \( v \in M \) and by (2.4), \( f \in L \).

**Lemma 2.3.** \( X \) may be decomposed uniquely into the disjoint union \( X = X_0 \cup X_1 \) where (i) \( X_0, X_1 \in \Sigma_i \). (ii) There exists a finite invariant measure for \( \Pi \) equivalent to \( m|_{X_1} \). (iii) There exists \( \{A_n\} \subset \Sigma \) with \( A_n \not\subset X_0 \) and \( I_{A_n} \in L, \forall n \).

**Proof.** Let \( \mu \) be any finite invariant measure for \( \Pi \), let \( B = \text{supp} \mu \); it is easy to see that \( B \in \Sigma_i \). Let \( \alpha = \sup \{m(B) \mid B = \text{supp} \mu, \mu \text{ a finite invariant measure for } \Pi \} \). Hence there exists a sequence of finite invariant measures \( \{\mu_n\} \), such that \( m(B_n) \not\geq \alpha \), where \( B_n = \text{sup} \mu_n \). Define \( X_1 = \bigcup_{n=1}^{\infty} B_n \) and \( \lambda = \sum_{n=1}^{\infty} (1/2^n) \mu_n \). Then there exists a finite invariant measure with \( \text{supp } \lambda = X_1 \), \( m(X_1) = \alpha \), and \( X_1 \in \Sigma_i \) (or \( P1x_1 = 1x_1, \forall P \in \Pi \)). Define \( X_0 = X - X_1 \), assume that there exists a finite invariant measure for \( \Pi \), \( \lambda' \) supported on \( X_0 \). Let \( \text{supp } \lambda' = B' \subset X_0 \), then \( \lambda + \lambda' \) is a finite invariant measure for \( \Pi \), \( \text{supp}(\lambda + \lambda') = X_1 \cup B' \) and \( m(X_1 \cup B') > \alpha \), a contradiction.
Since $X_0 \in \Sigma_i$ (and $P_1X_0 = 1_{X_0}$) we can restrict the Markov operators of $\Pi$ to $(X_0, \Sigma_{X_0}, m_{X_0})$, and apply Lemma 2.2. Formula (2.4) implies that if $0 \leq g \leq f$ and $f \in L$ and $g \in L$, and if $0 \leq f$, $g \in L$, then $\max(f, g) \in L$. Let $\mathcal{A} = \{A \mid A \in L\}$; clearly if $0 \leq f \in L$ and $A = \{f \geq \varepsilon > 0\}$ then $A \in \mathcal{A}$, if $A \in \mathcal{A}$ and $B \subseteq A$ then $B \in \mathcal{A}$, and if $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$. Let $\beta = \sup\{m(A) \mid A \in \mathcal{A}\}$. There exists a sequence $\{A_n\}$ such that $m(A_n) > \beta$. It can be supposed that $A_n \subseteq A_{n+1}$, $\forall n$ (if not, replace $A_n$ by $A' = \bigcup_{n=1}^\infty A_n$, and then $A' \in \mathcal{A}$ and $m(A') > \beta$). Let $A_n \not\subseteq X'_0$. If $X'_0 \not= X_0$, then there exists $P \in \Pi$ such that $\{P1_{X'_0} > 0\} \not\subseteq X_0$, and hence $\exists \varepsilon > 0$ such that $\{P1_{A_n} > \varepsilon\} \not\subseteq X'_0$. Let $E = \{P1_{A_n} > \varepsilon\}$, $1_{E} \in L$ by (2.4), because for each $v \in M$ we have:

$$\langle v, 1_{E} \rangle \leq (1/\varepsilon) \cdot \langle v, P1_{A_n} \rangle = (1/\varepsilon) \cdot \langle v, 1_{A_n} \rangle = 0.$$  

Denote $A'_n = A_n \cup E$, $A'_n \in \mathcal{A}$ and $m(A'_n) > m(X'_0 \cup E) > \beta$, a contradiction, hence $X'_0 \subseteq X_0$.

If $X'_0 \not= X_0$, then we can restrict the Markov operators of $\Pi$ to $(X_0 - X'_0, \Sigma_{X_0 - X'_0}, m_{X_0 - X'_0})$, and by Lemma 2.2 there exists $\mathcal{B} \subseteq E \subseteq X_0 - X'_0$, denote $A'_n' = A_n \cup E$, $A'_n' \in \mathcal{A}$ and $m(A'_n') > m(X'_0 \cup E) > \beta$, a contradiction. So, $X'_0 = X_0$ and Lemma 2.3 is proved.

**Lemma 2.4.** Let $\{A_n\}$ be the sequence of Lemma 2.3, part (iii), then for each $n$ and for each $\varepsilon > 0$, there exists an operator $Q \in \Pi$ such that $\|Q1_{A_n}\| < \varepsilon$.

**Proof.** $1_{A_n} \in L$, hence there exist $f_1, f_2, \cdots, f_j \in L_\infty$ and $P_1, P_2, \cdots, P_j \in \Pi$ such that

$$\|f_1 - P_1f_1\| + \|f_2 - P_2f_2\| + \cdots + \|f_j - P_jf_j\| < \varepsilon/2.$$  

Hence:

$$\left\| \frac{1}{N_j} \sum_{i=1}^{N} \sum_{i=1}^{N} \cdots P_1^{i_1}P_2^{i_2}\cdots P_j^{i_j}1_{A_n} \right\|_\infty \leq \left\| \frac{1}{N_j} \sum_{i=1}^{N} \sum_{i=1}^{N} \cdots P_1^{i_1}P_2^{i_2}\cdots P_j^{i_j}(f_1 - P_1f_1) \right\|_\infty + \cdots + \left\| \frac{1}{N_j} \sum_{i=1}^{N} \sum_{i=1}^{N} \cdots P_1^{i_1}P_2^{i_2}\cdots P_j^{i_j}(f_j - P_jf_j) \right\|_\infty + \cdots + \left\| \frac{1}{N_j} \sum_{i=1}^{N} \sum_{i=1}^{N} \cdots P_1^{i_1}P_2^{i_2}\cdots P_j^{i_j} \right\|_\infty \times \left[ (f_1 - P_1f_1) + (f_2 - P_2f_2) + \cdots + (f_j - P_jf_j) - 1_{A_n} \right]_\infty.$$
But
\[
\left\| \frac{1}{N^j} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_j=1}^{N} p_{1}^{i_1} p_{2}^{i_2} \cdots p_{k}^{i_j} (f_k - P_k f_k) \right\|_{\infty}
\]
\[
= \left\| \frac{1}{N^j} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_k=1}^{N} \sum_{i_{k+1}=1}^{N} \sum_{i_j=1}^{N} p_{1}^{i_1} p_{2}^{i_2} \cdots p_{k}^{i_k} \cdots p_{k}^{i_{k+1}} \cdots p_{j}^{i_j} \right\|_{\infty}
\times \left[ \frac{1}{N} \sum_{i_k=1}^{N} p_{k}^{i_k} (f_k - P_k f_k) \right]_{\infty}
\]
\[
\leq 2 \| f_k \|_{\infty}/N,
\]
if \( N \) is sufficiently large then \( 2 \| f_k \|_{\infty}/N \leq \varepsilon/2j \) for \( 1 \leq k \leq j \). Let
\[
Q = \frac{1}{N^j} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_j=1}^{N} p_{1}^{i_1} p_{2}^{i_2} \cdots p_{j}^{i_j}
\]
and then \( \| Q_1 f_{n} \| \leq \varepsilon \).

**Lemma 2.5.** Let \( \{ A_n \} \) be the sequence of Lemma 2.3, part (iii), then there exists an operator \( V \in \mathfrak{F} \) such that \( \lim_{k \to \infty} \| V^{k} A_{n} \| = 0 \), \( \forall n \).

**Proof.** By Lemma 2.4 it can be shown that there exists a sequence of operators \( \{ Q_{n} \} \subset \mathfrak{F} \) such that \( \| Q_{n} A_{n} \| = 1/n \). Let \( a_{n} \) be positive numbers such that \( \sum_{n=1}^{\infty} a_{n} = 1 \). Let \( V = \sum_{n=1}^{\infty} a_{n} Q_{n} \), then \( V \in \mathfrak{F} \). Given an integer \( N \), denote \( \beta = \sum_{n=1}^{N} a_{n} \), \( \gamma = \sum_{n=N+1}^{\infty} a_{n} \), \( \beta + \gamma = 1 \). Define the operators
\[
R = (1/\beta) \sum_{n=1}^{N} a_{n} Q_{n} \quad \text{and} \quad S = (1/\gamma) \sum_{n=N+1}^{\infty} a_{n} Q_{n},
\]
\( R \), \( S \in \mathfrak{F} \), \( \beta R + \gamma S = V \), and \( \| S^{1} A_{n} \| \leq 1/N \). Hence
\[
\| V^{k} A_{n} \| = \| (\beta R + \gamma S)^{k} A_{n} \| \leq \beta^{k} \| R^{k} A_{n} \| + \| S^{1} A_{n} \| \leq \beta^{k} + 1/N.
\]
Thus for \( k \) sufficiently large we have for each \( 1 \leq n \leq N \), \( \| V^{k} A_{n} \| \leq 2/N \) but \( N \) is arbitrary, hence \( \lim_{k \to \infty} \| V^{k} A_{n} \| = 0 \), \( \forall n \).

**Theorem.** Let \( X = X_{0} \cup X_{1} \) be as in Lemma 2.3. Then there exists an operator \( U \in \mathfrak{F} \) such that \( X_{1} \) and \( X_{0} \) are the conservative and dissipative parts, respectively, for \( U \).

**Proof.** Let \( V \) be as in Lemma 2.5. Define the sequence of integers \( \{ n_{k} \} \) inductively:
\[
n_{1} = 1, \quad n_{k+1} = n_{k} + 1, \quad \| V^{j} A_{n_{k+1}} \| \leq \frac{1}{n_{k} + 1}, \quad \forall j \geq k + 1,
\]
\[
= n_{k}, \quad \text{otherwise}.
\]
Clearly \( n_{k} \to \infty \), and \( \lim_{k \to \infty} \| V^{k} A_{n} \| = 0 \), where \( \{ A_{n} \} \) is the sequence of Lemma 2.3, part (iii). Let \( \{ c_{k} \} \) be the sequence of Lemma 3 of [1] such that \( \sum_{k=0}^{\infty} c_{k} \| V^{k} A_{n} \| < \infty \). It is obvious that \( \sum_{k=0}^{\infty} c_{k} \| V^{k} A_{n} \| < \infty \) for each \( n \).
Now let \( \{a_k\} \) be the sequence associated with \( \{c_k\} \) as in Lemma 2 of [1] and let \( U = \sum_{k=0}^{\infty} a_k V^k \), then \( U \in \Pi \) and the proof of Theorem 1 of [1] shows that \( \sum_{k=0}^{\infty} U^k A_n \in L_\infty \) for each \( n \), hence \( X_0 = \bigcup_{n=1}^{\infty} A_n \) is contained in the dissipative part with respect to \( U \). On the other hand if \( \mu \) is a finite invariant measure for \( \Pi \), supported on \( X_1 \), then \( \mu U = \mu \) and \( X_1 \) is contained in the conservative part with respect to \( U \).

**References**


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