

## ON FINITE INVARIANT MEASURES FOR SETS OF MARKOV OPERATORS

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**ABSTRACT.** A. Brunel [1] proved that a Markovian operator  $P$  has an invariant measure if and only if each convex combination of iterates  $\sum_{n=0}^{\infty} \alpha_n P^n$  is conservative. In the present paper this result is generalized for any commutative semigroup of Markovian operators: Let  $\Pi$  be a semigroup; there exists a *common* invariant measure for  $\Pi$  if and only if each convex combination  $\sum_{n=1}^{\infty} \alpha_n P_n$ , where  $\{P_n\} \subset \Pi$ , is conservative.

**1. Definitions and notations.** Let  $(X, \Sigma, m)$  be a finite measure space. A Markov operator  $P$  is a positive contraction on  $L_1(X, \Sigma, m)$  i.e., (i)  $\|P\| \leq 1$ , (ii)  $u \geq 0 \Rightarrow uP \geq 0$ . We shall use the notations of [3], so the operator adjoint to  $P$  which is defined in  $L_{\infty}(m)$  will also be denoted by  $P$  to the left side of the variable. Thus  $\langle uP, f \rangle = \langle u, Pf \rangle$ ,  $u \in L_1(m)$ ,  $f \in L_{\infty}(m)$ . We denote  $\Sigma_i(P) = \{A \in \Sigma \mid P1_A = 1_A \text{ a.e.}\}$ . If  $P$  is conservative, then  $\Sigma_i(P)$  is a field.

Let us consider the commutative semigroup  $\Pi$  of conservative Markov operators. The *invariant sets* of  $\Pi$  is the collection  $\Sigma_i = \bigcap_{P \in \Pi} \Sigma_i(P)$ .

Let  $\bar{\Pi}$  be the convex hull of  $\Pi$ : (i)  $\Pi \subset \bar{\Pi}$ , (ii)  $\{P_n\} \subset \bar{\Pi}$  and  $P_n \rightarrow P$  in operator norm then  $P \in \bar{\Pi}$ . (iii)  $P_1, P_2 \in \bar{\Pi}$ ,  $0 < \alpha, \beta < 1$  and  $\alpha + \beta = 1$  then  $\alpha P_1 + \beta P_2 \in \bar{\Pi}$ . (iv)  $\bar{\Pi}$  is minimal under those conditions.

A measure  $\mu$  is said to be *invariant* for  $\Pi$  if  $\mu P = \mu$  for every  $P \in \Pi$ . Clearly,  $\bar{\Pi}$  is a semigroup and if  $\mu$  is a finite invariant measure for  $\Pi$  it is invariant for  $\bar{\Pi}$ .

In this paper we prove that if there is no finite invariant measure for  $\Pi$  then there exists an operator  $Q \in \bar{\Pi}$  such that the dissipative part of  $Q$ ,  $D = X$ . (For definitions see [3].)

**REMARK.** In [1] it is proved for  $\Pi = \{P^n\}$ , where  $P$  is a given Markov operator, that if it has no finite invariant measures, then there exists  $Q \in \bar{\Pi}$ , such that the dissipative part of  $Q$ ,  $D \neq \emptyset$ .

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## 2. Conservative operators and invariant measures.

LEMMA 2.1. *If there are no finite invariant measures for  $\Pi$ , then there exists  $\{P_n\} \subset \Pi$  and  $0 \leq f \in L_\infty(m)$  such that  $\sum_{n=1}^\infty P_n f \in L_\infty(m)$ .*

PROOF. In [4] it is proved that if there exists no finite invariant measure then there exists  $0 \leq g \in L_\infty(m)$  such that  $\inf\{\int P g \, dm \mid P \in \Pi\} = 0$ . Hence by slight modifications of the proof of Lemma C, Chapter IV of [3], or of the more elegant proof of this lemma which appears in [2], Lemma 2.1 can be proved.

Let us define the space:

$$(2.1) \quad L = \text{spn}\{(I - P)L_\infty(m) \mid P \in \Pi\}.$$

The orthogonal complement of  $L$  is

$$(2.2) \quad L^\perp = \{\nu \in L_\infty^*(m) \mid \nu P = \nu \, \forall P \in \Pi\}.$$

$L_\infty^*(m)$  is the space of the charges (finitely additive measures). If  $P1=1$ , as it is in the conservative case, then  $\nu P = \nu$  implies  $\nu^+ P = \nu^+$ . Define:

$$(2.3) \quad M = \{\nu \in L_\infty^*(m) \mid \nu \geq 0, \|\nu\| = 1, \nu P = \nu \, \forall P \in \Pi\}.$$

It is easy to show that:

$$(2.4) \quad f \in L \Leftrightarrow \langle \nu, f \rangle = 0, \quad \nu \in M.$$

LEMMA 2.2. *If there exists no finite invariant measures for  $\Pi$ , then there exists  $0 \neq f \geq 0$  such that  $f \in L$ .*

PROOF. Let  $f$  be the function of Lemma 2.1. It is clear that  $\langle \nu, f \rangle = 0$  or each  $\nu \in M$  and by (2.4),  $f \in L$ .

LEMMA 2.3.  *$X$  may be decomposed uniquely into the disjoint union  $X = X_0 \cup X_1$  where (i)  $X_0, X_1 \in \Sigma_i$ . (ii) There exists a finite invariant measure for  $\Pi$  equivalent to  $m|_{X_1}$ . (iii) There exists  $\{A_n\} \subset \Sigma$  with  $A_n \nearrow X_0$ , and  $1_{A_n} \in L, \forall n$ .*

PROOF. Let  $\mu$  be any finite invariant measure for  $\Pi$ , let  $B = \text{supp } \mu$ ; it is easy to see that  $B \in \Sigma_i$ . Let  $\alpha = \sup\{m(B) \mid B = \text{supp } \mu, \mu \text{ a finite invariant measure for } \Pi\}$ . Hence there exists a sequence of finite invariant measures  $\{\mu_n\}$ , such that  $m(B_n) \nearrow \alpha$ , where  $B_n = \text{supp } \mu_n$ . Define  $X_1 = \bigcup_{n=1}^\infty B_n$  and  $\lambda = \sum_{n=1}^\infty (1/2^n)\mu_n$  and then  $\lambda$  is a finite invariant measure with  $\text{supp } \lambda = X_1$ ,  $m(X_1) = \alpha$ , and  $X_1 \in \Sigma_i$  (or  $P1_{X_1} = 1_{X_1}, \forall P \in \Pi$ ). Define  $X_0 = X - X_1$ , assume that there exists a finite invariant measure for  $\Pi$ ,  $\lambda'$  supported on  $X_0$ . Let  $\text{supp } \lambda' = B' \subset X_0$ , then  $\lambda + \lambda'$  is a finite invariant measure for  $\Pi$ ,  $\text{supp}(\lambda + \lambda') = X_1 \cup B'$  and  $m(X_1 \cup B') > \alpha$ , a contradiction.

Since  $X_0 \in \Sigma_i$  (and  $P1_{X_0} = 1_{X_0}$ ) we can restrict the Markov operators of  $\Pi$  to  $(X_0, \Sigma_{X_0}, mI_{X_0})$ , and apply Lemma 2.2. Formula (2.4) implies that if  $0 \leq g \leq f$  and  $f \in L$  and  $g \in L$ , and if  $0 \leq f, g \in L$ , then  $\max(f, g) \in L$ . Let  $\mathfrak{A} = \{A | 1_A \in L\}$ ; clearly if  $0 \leq f \in L$  and  $A = \{f \geq \varepsilon > 0\}$  then  $A \in \mathfrak{A}$ , if  $A \in \mathfrak{A}$  and  $B \subset A$  then  $B \in \mathfrak{A}$ , and if  $A, B \in \mathfrak{A}$  then  $A \cup B \in \mathfrak{A}$ . Let  $\beta = \sup\{m(A) | A \in \mathfrak{A}\}$ . There exists a sequence  $\{A_n\}$  such that  $m(A_n) \nearrow \beta$ . It can be supposed that  $A_n \subset A_{n+1}, \forall_n$  (if not, replace  $A_n$  by  $A'_n = \bigcup_{k=1}^n A_k$ , and then  $A'_n \in \mathfrak{A}$  and  $m(A'_n) \nearrow \beta$ ). Let  $A_n \nearrow X'_0$ . If  $X'_0 \notin \Sigma_i$  then there exists  $P \in \Pi$  such that  $\{P1_{X'_0} > 0\} \not\subset X'_0$ , and hence  $\exists_n, \varepsilon > 0$  such that  $\{P1_{A_n} > \varepsilon\} \not\subset X'_0$ . Let  $E = \{P1_{A_n} > \varepsilon\}, 1_E \in L$  by (2.4), because for each  $v \in M$  we have:

$$\langle v, 1_E \rangle \leq (1/\varepsilon) \cdot \langle v, P1_{A_n} \rangle = (1/\varepsilon) \cdot \langle v, 1_{A_n} \rangle = 0.$$

Denote  $A'_n = A_n \cup E, A'_n \in \mathfrak{A}$  and  $m(A'_n) \nearrow m(X'_0 \cup E) > \beta$ , a contradiction, hence  $X'_0 \in \Sigma_i$ .

If  $X'_0 \neq X_0$ , then we can restrict the Markov operators of  $\Pi$  to  $(X_0 - X'_0, \Sigma_{X_0 - X'_0}, mI_{X_0 - X'_0})$ , and by Lemma 2.2 there exists  $\mathfrak{A} \ni E \subset X_0 - X'_0$ , denote  $A'_n = A_n \cup E, A'_n \in \mathfrak{A}$  and  $m(A'_n) \nearrow m(X'_0 \cup E) > \beta$ , a contradiction. So,  $X'_0 = X_0$  and Lemma 2.3 is proved.

LEMMA 2.4. *Let  $\{A_n\}$  be the sequence of Lemma 2.3, part (iii), then for each  $n$  and for each  $\varepsilon > 0$ , there exists an operator  $Q \in \Pi$  such that  $\|Q1_{A_n}\|_\infty < \varepsilon$ .*

PROOF.  $1_{A_n} \in L$ , hence there exist  $f_1, f_2, \dots, f_j \in L_\infty$  and  $P_1, P_2, \dots, P_j \in \Pi$  such that

$$\|(f_1 - P_1 f_1) + (f_2 - P_2 f_2) + \dots + (f_j - P_j f_j) - 1_{A_n}\| < \varepsilon/2.$$

Hence:

$$\begin{aligned} & \left\| \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \dots \sum_{i_j=1}^N P_1^{i_1} P_2^{i_2} \dots P_j^{i_j} 1_{A_n} \right\|_\infty \\ & \leq \left\| \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \dots \sum_{i_j=1}^N P_1^{i_1} P_2^{i_2} \dots P_j^{i_j} (f_1 - P_1 f_1) \right\|_\infty \\ & \quad + \left\| \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \dots \sum_{i_j=1}^N P_1^{i_1} P_2^{i_2} \dots P_j^{i_j} (f_2 - P_2 f_2) \right\|_\infty + \dots \\ & \quad + \left\| \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \dots \sum_{i_j=1}^N P_1^{i_1} P_2^{i_2} \dots P_j^{i_j} (f_j - P_j f_j) \right\|_\infty \\ & \quad + \left\| \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \dots \sum_{i_j=1}^N P_1^{i_1} P_2^{i_2} \dots P_j^{i_j} \right. \\ & \quad \left. \times [(f_1 - P_1 f_1) + (f_2 - P_2 f_2) + \dots + (f_j - P_j f_j) - 1_{A_n}] \right\|_\infty. \end{aligned}$$

But

$$\begin{aligned} & \left\| \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_k=1}^N \cdots \sum_{i_j=1}^N P_1^{i_1} P_2^{i_2} \cdots P_k^{i_k} \cdots P_j^{i_j} (f_k - P_k f_k) \right\|_{\infty} \\ &= \left\| \frac{1}{N^{j-1}} \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_{k-1}=1}^N \sum_{i_{k+1}=1}^N \cdots \sum_{i_j=1}^N P_1^{i_1} P_2^{i_2} \cdots P_{k-1}^{i_{k-1}} P_{k+1}^{i_{k+1}} \cdots P_j^{i_j} \right. \\ & \quad \left. \times \left[ \frac{1}{N} \sum_{i_k=1}^N P_k^{i_k} (f_k - P_k f_k) \right] \right\|_{\infty} \\ & \leq 2 \|f_k\|_{\infty} / N, \end{aligned}$$

if  $N$  is sufficiently large then  $2 \|f_k\|_{\infty} / N \leq \varepsilon / 2j$  for  $1 \leq k \leq j$ . Let

$$Q = \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_j=1}^N P_1^{i_1} P_2^{i_2} \cdots P_j^{i_j}$$

and then  $\|Q 1_{A_n}\| < \varepsilon$ .

LEMMA 2.5. Let  $\{A_n\}$  be the sequence of Lemma 2.3, part (iii), then there exists an operator  $V \in \bar{\Pi}$  such that  $\lim_{k \rightarrow \infty} \|V^k 1_{A_{n_k}}\|_{\infty} = 0, \forall n$ .

PROOF. By Lemma 2.4 it can be shown that there exists a sequence of operators  $\{Q_n\} \subset \bar{\Pi}$  such that  $\|Q_n 1_{A_n}\| \leq 1/n$ . Let  $\alpha_n$  be positive numbers such that  $\sum_{n=1}^{\infty} \alpha_n = 1$ . Let  $V = \sum_{n=1}^{\infty} \alpha_n Q_n$ , then  $V \in \bar{\Pi}$ . Given an integer  $N$ , denote  $\beta = \sum_{n=1}^N \alpha_n, \gamma = \sum_{n=N+1}^{\infty} \alpha_n, \beta + \gamma = 1$ . Define the operators  $R = (1/\beta) \sum_{n=1}^N \alpha_n Q_n$  and  $S = (1/\gamma) \sum_{n=N+1}^{\infty} \alpha_n Q_n, R, S \in \bar{\Pi}, \beta R + \gamma S = V$ , and  $\|S 1_{A_N}\|_{\infty} \leq 1/N$ . Hence

$$\|V^k 1_{A_N}\|_{\infty} = \|(\beta R + \gamma S)^k 1_{A_N}\|_{\infty} \leq \beta^k \|R^k 1_{A_N}\|_{\infty} + \|S 1_{A_N}\|_{\infty} \leq \beta^k + 1/N.$$

Thus for  $k$  sufficiently large we have for each  $1 \leq n \leq N, \|V^k 1_{A_n}\|_{\infty} \leq 2/N$  but  $N$  is arbitrary, hence  $\lim_{k \rightarrow \infty} \|V^k 1_{A_n}\|_{\infty} = 0, \forall n$ .

THEOREM. Let  $X = X_0 \cup X_1$  be as in Lemma 2.3. Then there exists an operator  $U \in \bar{\Pi}$  such that  $X_1$  and  $X_0$  are the conservative and dissipative parts, respectively, for  $U$ .

PROOF. Let  $V$  be as in Lemma 2.5. Define the sequence of integers  $\{n_k\}$  inductively:

$$\begin{aligned} n_1 = 1, \quad n_{k+1} = n_k + 1, \quad \|V^j 1_{A_{n_{k+1}}}\|_{\infty} &\leq \frac{1}{n_k + 1}, & \forall j \geq k + 1, \\ &= n_k, & \text{otherwise.} \end{aligned}$$

Clearly  $n_k \nearrow \infty$ , and  $\lim_{k \rightarrow \infty} \|V^k 1_{A_{n_k}}\| = 0$ , where  $\{A_n\}$  is the sequence of Lemma 2.3, part (iii). Let  $\{c_k\}$  be the sequence of Lemma 3 of [1] such that  $\sum_{k=0}^{\infty} c_k \|V^k 1_{A_{n_k}}\|_{\infty} < \infty$ . It is obvious that  $\sum_{k=0}^{\infty} c_k \|V^k 1_{A_n}\|_{\infty} < \infty$  for each  $n$ .

Now let  $\{a_k\}$  be the sequence associated with  $\{c_k\}$  as in Lemma 2 of [1] and let  $U = \sum_{k=0}^{\infty} a_k V^k$ , then  $U \in \Pi$  and the proof of Theorem 1 of [1] shows that  $\sum_{k=0}^{\infty} U^k 1_{A_n} \in L_{\infty}$  for each  $n$ , hence  $X_0 = \bigcup_{n=1}^{\infty} A_n$  is contained in the dissipative part with respect to  $U$ . On the other hand if  $\mu$  is a finite invariant measure for  $\Pi$ , supported on  $X_1$ , then  $\mu U = \mu$  and  $X_1$  is contained in the conservative part with respect to  $U$ .

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