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ON FINITE INVARIANT MEASURES FOR SETS OF MARKOV OPERATORS

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ABSTRACT. A. Brunel [1] proved that a Markovian operator P has an invariant measure if and only if each convex combination of iterates $\sum_{n=0}^{\infty} \alpha_n P^n$ is conservative. In the present paper this result is generalized for any commutative semigroup of Markovian operators: Let Π be a semigroup; there exists a *common* invariant measure for Π if and only if each convex combination $\sum_{n=1}^{\infty} \alpha_n P_n$, where $\{P_n\} \subset \Pi$, is conservative.

1. **Definitions and notations.** Let (X, Σ, m) be a finite measure space. A Markov operator P is a positive contraction on $L_1(X, \Sigma, m)$ i.e., (i) $\|P\| \leqq 1$, (ii) $u \geqq 0 \Rightarrow uP \geqq 0$. We shall use the notations of [3], so the operator adjoint to P which is defined in $L_\infty(m)$ will also be denoted by P to the left side of the variable. Thus $\langle uP, f \rangle = \langle u, Pf \rangle$, $u \in L_1(m)$, $f \in L_\infty(m)$. We denote $\Sigma_i(P) = \{A \in \Sigma \mid P1_A = 1_A \text{ a.e.}\}$. If P is conservative, then $\Sigma_i(P)$ is a field.

Let us consider the commutative semigroup Π of conservative Markov operators. The *invariant sets* of Π is the collection $\Sigma_i = \bigcap_{P \in \Pi} \Sigma_i(P)$.

Let $\bar{\Pi}$ be the convex hull of Π: (i) $\Pi \subset \bar{\Pi}$, (ii) $\{P_n\} \subset \bar{\Pi}$ and $P_n \to P$ in operator norm then $P \in \Pi$. (iii) $P_1, P_2 \in \bar{\Pi}$, $0 < a, \beta < 1$ and $\alpha + \beta = 1$ then $\alpha P_1 + \beta P_2 \in \bar{\Pi}$. (iv) $\bar{\Pi}$ is minimal under those conditions.

A measure μ is said to be *invariant* for Π if $\mu P = \mu$ for every $P \in \Pi$. Clearly, $\bar{\Pi}$ is a semigroup and if μ is a finite invariant measure for Π it is invariant for $\bar{\Pi}$.

In this paper we prove that if there is no finite invariant measure for Π then there exists an operator $Q \in \bar{\Pi}$ such that the dissipative part of Q, $D = X$. (For definitions see [3].)

REMARK. In [1] it is proved for $\Pi = \{P^n\}$, where P is a given Markov operator, that if it has no finite invariant measures, then there exists $Q \in \bar{\Pi}$, such that the dissipative part of Q, $D \neq \varnothing$.

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2. Conservative operators and invariant measures.

LEMMA 2.1. *If there are no finite invariant measures for* Π, *then there exists* $\{P_n\}\subset\Pi$ *and* $0\leqq f\in L_\infty(m)$ *such that* $\sum_{n=1}^{\infty} P_n f\in L_\infty(m)$.

PROOF. In [4] it is proved that if there exists no finite invariant measure then there exists $0\leqq g\in L_\infty(m)$ such that $\inf\{\int Pg\,dm|P\in\Pi\}=0$. Hence by slight modifications of the proof of Lemma C, Chapter IV of [3], or of the more elegant proof of this lemma which appears in [2], Lemma 2.1 can be proved.

Let us define the space:

$$L=\operatorname{spn}\{(I-P)L_\infty(m)\mid P\in\Pi\}. \tag{2.1}$$

The orthogonal complement of L is

$$L^\perp=\{\nu\in L^*_\infty(m)\mid \nu P=\nu\ \forall P\in\Pi\}. \tag{2.2}$$

$L^*_\infty(m)$ is the space of the charges (finitely additive measures). If $P1=1$, as it is in the conservative case, then $\nu P=\nu$ implies $\nu^+P=\nu^+$. Define:

$$M=\{\nu\in L^*_\infty(m)\mid \nu\geqq 0,\ \|\nu\|=1,\ \nu P=\nu\ \forall P\in\Pi\}. \tag{2.3}$$

It is easy to show that:

$$f\in L\Leftrightarrow\langle v,f\rangle=0,\qquad \nu\in M. \tag{2.4}$$

LEMMA 2.2. *If there exists no finite invariant measures for* Π, *then there exists* $0\neq f\geqq 0$ *such that* $f\in L$.

PROOF. Let f be the function of Lemma 2.1. It is clear that $\langle\nu,f\rangle=0$ or each $\nu\in M$ and by (2.4), $f\in L$. .

LEMMA 2.3. *X may be decomposed uniquely into the disjoint union* $X=X_0\cup X_1$ *where* (i) $X_0, X_1\in\Sigma_i$. (ii) *There exists a finite invariant measure for* Π *equivalent to* mI_{X_1}. (iii) *There exists* $\{A_n\}\subset\Sigma$ *with* $A_n\nearrow X_0$, *and* $1_{A_n}\in L$, $\forall_n$.

PROOF. Let μ be any finite invariant measure for Π, let $B=\operatorname{supp}\mu$; it is easy to see that $B\in\Sigma_i$. Let $\alpha=\sup\{m(B)|B=\operatorname{supp}\mu,\ \mu$ a finite invariant measure for $\Pi\}$. Hence there exists a sequence of finite invariant measures $\{\mu_n\}$, such that $m(B_n)\nearrow\alpha_\infty$, where $B_n=\sup\mu_n$. Define $X_1=\bigcup_{n=1}^\infty B_n$ and $\lambda=\sum_{n=1}^\infty(1/2^n)\mu_n$ and then λ is a finite invariant measure with $\operatorname{supp}\lambda=x_1$, $m(X_1)=\alpha$, and $X_1\in\Sigma_i$ (or $P1_{X_1}=1_{X_1}$, $\forall P\in\Pi$). Define $X_0=X-X_1$, assume that there exists a finite invariant measure for Π, λ' supported on X_0. Let $\operatorname{supp}\lambda'=B'\subset X_0$, then $\lambda+\lambda'$ is a finite invariant measure for Π, $\operatorname{supp}(\lambda+\lambda')=X_1\cup B'$ and $m(X_1\cup B')>\alpha$, a contradiction.

Since $X_0 \in \Sigma_i$ (and $P1_{X_0}=1_{X_0}$) we can restrict the Markov operators of Π to $(X_0, \Sigma_{X_0}, mI_{X_0})$, and apply Lemma 2.2. Formula (2.4) implies that if $0 \leqq g \leqq f$ and $f \in L$ and $g \in L$, and if $0 \leqq f$, $g \in L$, then $\max(f, g) \in L$. Let $\mathfrak{A}=\{A \mid 1_A \in L\}$; clearly if $0 \leqq f \in L$ and $A=\{f \geqq \varepsilon>0\}$ then $A \in \mathfrak{A}$, if $A \in \mathfrak{A}$ and $B \subset A$ then $B \in \mathfrak{A}$, and if A, $B \in \mathfrak{A}$ then $A \cup B \in \mathfrak{A}$. Let $\beta=\sup\{m(A) \mid A \in \mathfrak{A}\}$. There exists a sequence $\{A_n\}$ such that $m(A_n) \nearrow \beta$. It can be supposed that $A_n \subset A_{n+1}$, $\forall_n$ (if not, replace A_n by $A'_n=\bigcup_{k=1}^n A_n$, and then $A'_n \in \mathfrak{A}$ and $m(A'_n) \nearrow \beta$). Let $A_n \nearrow X'_0$. If $X'_0 \notin \Sigma_i$ then there exists $P \in \Pi$ such that $\{P1_{X_0'}>0\} \not\subset X'_0$, and hence $\exists_n$, $\varepsilon>0$ such that $\{P1_{A_n}>\varepsilon\} \not\subset X'_0$. Let $E=\{P1_{A_n}>\varepsilon\}$, $1_E \in L$ by (2.4), because for each $\nu \in M$ we have:

$$\langle \nu, 1_B \rangle \leqq (1/\varepsilon) \cdot \langle \nu, P1_{A_n} \rangle = (1/\varepsilon) \cdot \langle \nu, 1_{A_n} \rangle = 0.$$

Denote $A'_n=A_n \cup E$, $A'_n \in \mathfrak{A}$ and $m(A'_n) \nearrow m(X'_0 \cup E)>\beta$, a contradiction, hence $X'_0 \in \Sigma_i$.

If $X'_0 \neq X_0$, then we can restrict the Markov operators of Π to $(X_0-X'_0$, $\Sigma_{X_0-X_0'}$, $mI_{X_0-X_0'})$, and by Lemma 2.2 there exists $\mathfrak{A} \ni E \subset X_0-X'_0$, denote $A'_n=A_n \cup E$, $A'_n \in \mathfrak{A}$ and $m(A'_n) \nearrow m(X'_0 \cup E)>\beta$, a contradiction. So, $X'_0=X_0$ and Lemma 2.3 is proved.

LEMMA 2.4. *Let $\{A_n\}$ be the sequence of Lemma* 2.3, *part* (iii), *then for each n and for each $\varepsilon>0$, there exists an operator $Q \in \Pi$ such that $\|Q1_{A_n}\|_\infty<\varepsilon$.*

PROOF. $1_{A_n} \in L$, hence there exist $f_1, f_2, \cdots, f_j \in L_\infty$ and $P_1, P_2, \cdots, P_j \in \Pi$ such that

$$\|(f_1-P_1f_1)+(f_2-P_2f_2)+\cdots+(f_j-P_jf_j)-1_{A_n}\| < \varepsilon/2.$$

Hence:

$$\begin{aligned}
&\left\| \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_j=1}^N P_1^{i_1}P_2^{i_2}\cdots P_j^{i_j} 1_{A_n} \right\|_\infty \\
&\quad \leqq \left\| \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_j=1}^N P_1^{i_1}P_2^{i_2}\cdots P_j^{i_j}(f_1-P_1f_1) \right\|_\infty \\
&\qquad + \left\| \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_j=1}^N P_1^{i_1}P_2^{i_2}\cdots P_j^{i_j}(f_2-P_2f_2) \right\|_\infty + \cdots \\
&\qquad + \left\| \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_j=1}^N P_1^{i_1}P_2^{i_2}\cdots P_j^{i_j}(f_j-P_jf_j) \right\|_\infty \\
&\qquad + \left\| \frac{1}{N^j} \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_j=1}^N P_1^{i_1}P_2^{i_2}\cdots P_j^{i_j} \right. \\
&\qquad\qquad \left. \times [(f_1-P_1f_1)+(f_2-P_2f_2)+\cdots+(f_j-P_jf_j)-1_{A_n}] \right\|_\infty .
\end{aligned}$$

But

$$\left\| \frac{1}{N^j} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_k=1}^{N} \cdots \sum_{i_j=1}^{N} P_1^{i_1} P_2^{i_2} \cdots P_k^{i_k} \cdots P_j^{i_j} (f_k - P_k f_k) \right\|_\infty$$
$$= \left\| \frac{1}{N^{j-1}} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_{k-1}=1}^{N} \sum_{i_{k+1}=1}^{N} \cdots \sum_{i_j=1}^{N} P_1^{i_1} P_2^{i_2} \cdots P_{k-1}^{i_{k-1}} P_{k+1}^{i_{k+1}} \cdots P_j^{i_k} \times \left[\frac{1}{N} \sum_{i_k=1}^{N} P_k^{i_k} (f_k - P_k f_k) \right] \right\|_\infty$$
$$\leqq 2\|f_k\|_\infty / N,$$

if N is sufficiently large then $2\|f_k\|_\infty/N \leqq \varepsilon/2j$ for $1 \leqq k \leqq j$. Let

$$Q = \frac{1}{N^j} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_j=1}^{N} P_1^{i_1} P_2^{i_2} \cdots P_j^{i_j}$$

and then $\|Q 1_{A_n}\| < \varepsilon$.

LEMMA 2.5. *Let $\{A_n\}$ be the sequence of Lemma* 2.3, *part* (iii), *then there exists an operator $V \in \Pi$ such that* $\lim_{k\to\infty} \|V^k l_{A_{n_k}}\|_\infty = 0$, $\forall n$.

PROOF. By Lemma 2.4 it can be shown that there exists a sequence of operators $\{Q_n\} \subset \Pi$ such that $\|Q_n 1_{A_n}\| \leqq 1/n$. Let α_n be positive numbers such that $\sum_{n=1}^\infty \alpha_n = 1$. Let $V = \sum_{n=1}^\infty \alpha_n Q_n$, then $V \in \Pi$. Givèn an integer N, denote $\beta = \sum_{n=1}^N \alpha_n$, $\gamma = \sum_{n=N+1}^\infty \alpha_n$, $\beta + \gamma = 1$. Define the operators $R = (1/\beta) \sum_{n=1}^N \alpha_n Q_n$ and $S = (1/\gamma) \sum_{n=N+1}^\infty \alpha_n Q_n$, R, $S \in \Pi$, $\beta R + \gamma S = V$, and $\|S 1_{A_N}\|_\infty \leqq 1/N$. Hence

$$\|V^k 1_{A_N}\|_\infty = \|(\beta R + \gamma S)^k 1_{A_N}\| \leqq \beta^k \|R^k 1_{A_N}\|_\infty + \|S 1_{A_N}\|_\infty \leqq \beta^k + 1/N.$$

Thus for k sufficiently large we have for each $1 \leqq n \leqq N$, $\|V^k 1_{A_n}\|_\infty \leqq 2/N$ but N is arbitrary, hence $\lim_{k\to\infty} \|V^k 1_{A_n}\|_\infty = 0$, $\forall n$.

THEOREM. *Let $X = X_0 \cup X_1$ be as in Lemma* 2.3. *Then there exists an operator $U \in \Pi$ such that X_1 and X_0 are the conservative and dissipative parts, respectively, for U.*

PROOF. Let V be as in Lemma 2.5. Define the sequence of integers $\{n_k\}$ inductively:

$$n_1 = 1, \qquad \begin{aligned} n_{k+1} &= n_k + 1, \quad \|V^j 1_{A_{n_{k+1}}}\|_\infty \leqq \frac{1}{n_k + 1}, && \forall j \geqq k+1, \\ &= n_k, && \text{otherwise.} \end{aligned}$$

Clearly $n_k \nearrow \infty$, and $\lim_{k\to\infty} \|V^k 1_{A_{n_k}}\| = 0$, where $\{A_n\}$ is the sequence of Lemma 2.3, part (iii). Let $\{c_k\}$ be the sequence of Lemma 3 of **[1]** such that $\sum_{k=0}^\infty c_k \|V^k 1_{A_{n_k}}\|_\infty < \infty$. It is obvious that $\sum_{k=0}^\infty c_k \|V^k 1_{A_n}\|_\infty < \infty$ for each n.

Now let $\{a_k\}$ be the sequence associated with $\{c_k\}$ as in Lemma 2 of [**1**] and let $U=\sum_{k=0}^{\infty} a_k V^k$, then $U \in \Pi$ and the proof of Theorem 1 of [**1**] shows that $\sum_{k=0}^{\infty} U^k 1_{A_n} \in L_\infty$ for each n, hence $X_0=\bigcup_{n=1}^{\infty} A_n$ is contained in the dissipative part with respect to U. On the other hand if μ is a finite invariant measure for Π, supported on X_1, then $\mu U=\mu$ and X_1 is contained in the conservative part with respect to U.

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