

ON A CONJECTURE OF A. J. HOFFMAN. II

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ABSTRACT. It is proved that certain incidence relations of hyperplanes and closed convex sets in a d -polytope can be preserved while replacing these sets by suitable polytopal subsets.

The purpose of this paper is to prove

THEOREM 1. *If P is a d -polytope in E^d and C_1, \dots, C_k are closed convex subsets of P , such that every hyperplane that meets P meets $\bigcup_{i=1}^k C_i$, then there exist polytopes D_1, \dots, D_k with $D_i \subseteq C_i$ for all $1 \leq i \leq k$, such that every hyperplane that meets P meets $\bigcup_{i=1}^k D_i$.*

This settles all the cases $(d, d-1, k)$, for all $d \geq 2$ and $k \geq 1$, of the following conjecture due to A. J. Hoffman [3]:

Conjecture (d, t, k) . If P is a d -polytope in E^d , $d \geq 1$; $t \geq 0$ and $k \geq 1$ are integers, C_1, \dots, C_k are closed convex subsets of P such that every (affine) t -flat that meets P meets $\bigcup_{i=1}^k C_i$; then there are polytopes D_1, \dots, D_k with $D_i \subseteq C_i$ for all $1 \leq i \leq k$, such that every t -flat that meets P meets $\bigcup_{i=1}^k D_i$.

A. J. Hoffman proved [3] conjecture $(d, 0, k)$, for all $d \geq 1$ and $k \geq 1$; in these cases the t -flats are points and C_1, \dots, C_k cover P .

It follows quite elementarily that conjecture $(d, t, 1)$ is true for all $d \geq 1$ and $t \geq 0$, since in this case $C_1 = P$ (see Remark 1, here). W. R. Hare, Jr. and C. R. Smith proved [2] that conjecture $(d, t, 2)$ is true for all $d \geq 1$ and $t \geq 0$. We have previously shown [4] that conjecture $(d, d-2, k)$ is false for all $d \geq 3$ and $k \geq 4$, while here it is shown that conjecture $(d, d-1, k)$ is true for all $d \geq 2$ and $k \geq 1$. Conjecture $(3, 1, 3)$ is true (see Remark 4).

DEFINITIONS. A *polytope* P is the convex hull of a finite set of points in the Euclidean d -dimensional space E^d ; a *d -polytope* in E^d is a polytope with nonempty interior; $\text{Vert } P$ denotes here the set of vertices of a polytope P ; if $A \subset E^d$, $\text{conv } A$ denotes the convex hull of A .

An (affine) *t -flat* in E^d is a translate of a t -dimensional subspace of E^d , and a *hyperplane* is a $(d-1)$ -flat. If H is a hyperplane in E^d , then H_+ and H_- ($\overset{\circ}{H}_+$ and $\overset{\circ}{H}_-$) denote the two closed (open, respectively) half-spaces of E^d , determined by H .

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A hyperplane H supports a compact set X in E^d if $X \subset H_+$ and $H \cap X \neq \emptyset$. A hyperplane H separates (strictly separates) A and B if $A \subset H_+$ and $B \subset H_-$ ($A \subset \mathring{H}_+$ and $B \subset \mathring{H}_-$, respectively). For additional definitions and information the reader is referred to [1].

REMARK 1. If P, C_1, \dots, C_k are as given in conjecture (d, t, k) , for some $d \geq 1, k \geq 1$ and $0 \leq t \leq d-1$, then $\text{Vert } P \subseteq \bigcup_{i=1}^k C_i$. To establish this, let $v \in \text{Vert } P$; there exists a hyperplane H_v of P such that $H_v \cap P = \{v\}$; H_v clearly contains a t -flat F_v through v . Since F_v is a t -flat that meets P (at v), it follows by the assumptions that F_v meets $\bigcup_{i=1}^k C_i$, i.e. $F_v \cap (\bigcup_{i=1}^k C_i) \neq \emptyset$, and since $\bigcup_{i=1}^k C_i \subset P, F_v \cap (\bigcup_{i=1}^k C_i) \subset F_v \cap P = \{v\}$; hence $v \in \bigcup_{i=1}^k C_i$ and therefore $\text{Vert } P \subseteq \bigcup_{i=1}^k C_i$.

In case $k=1$, it follows that $\text{Vert } P \subseteq C_1$, and therefore $C_1 = P$, since C_1 is a convex set contained in P . As a result, conjecture $(d, t, 1)$ is (trivially) true for all $d \geq 1$ and $0 \leq t \leq d-1$.

We need the following:

LEMMA 1. If C is a convex set in $E^d, x \in E^d$, and A and B are such that $C = \text{conv}(A \cup B)$, then every hyperplane that meets C meets $\text{conv}(A \cup x) \cup \text{conv}(B \cup x)$.

PROOF. Let Y be defined by $Y = \text{conv}(A \cup x) \cup \text{conv}(B \cup x)$; every pair of points y_1 and y_2 of Y are connected by the polyhedral path $y_1 x \cup x y_2$ which lies entirely in Y .

Let H be an arbitrary hyperplane such that $H \cap C \neq \emptyset$. If $A \subset \mathring{H}_+$ and $B \subset \mathring{H}_+$, then $\text{conv}(A \cup B) \subset \mathring{H}_+$, hence $C \subset \mathring{H}_+$, but this implies $C \cap H = \emptyset$ —a contradiction; therefore $A \cup B \not\subset H_+$, and similarly $A \cup B \not\subset H_-$. Therefore $A \cup B$ contains a point y_1 in H_+ and a point y_2 in H_- ; $y_1, y_2 \in Y$ since $Y \supset A \cup B$. The polyhedral path in Y that connects y_1 to y_2 clearly meets H , hence $Y \cap H \neq \emptyset$ and the proof is complete.

COROLLARY 1. If P is a polytope in $E^d, x \in E^d$, and $\text{Vert } P = A \cup B$, then every hyperplane that meets P meets $\text{conv}(A \cup x) \cup \text{conv}(B \cup x)$.

PROOF. In this case $P = \text{conv}(\text{Vert } P) = \text{conv}(A \cup B)$, and Lemma 1 is applicable (with $P = C$).

Since the replacement of k compact convex sets by polytopes can be done one at a time, we state and prove the following:

THEOREM 2. If P is a d -polytope in E^d, C_1, \dots, C_k are closed convex subsets of P such that every hyperplane that meets P meets $\bigcup_{i=1}^k C_i$, then there exists a polytope D_1 in C_1 such that every hyperplane that meets P meets $D_1 \cup \bigcup_{i=2}^k C_i$.

THEOREM 3. If P is a d -polytope in E^d, C_1, \dots, C_k are closed convex subsets of $P, C_i \cap C_j = \emptyset$ for all $1 \leq i < j \leq k$ and every hyperplane that meets

P meets $\bigcup_{i=1}^k C_i$, then there exists a polytope D_1 in C_1 such that every hyperplane that meets P meets $D_1 \cup \bigcup_{i=2}^k C_i$.

Clearly Theorems 1 and 2 are equivalent; our proof of Theorem 1 uses the following.

Claim 1. Theorem 3 implies Theorem 2.

PROOF OF CLAIM 1. Assuming Theorem 3 is true, we prove Theorem 2 by induction on k . In Case $k=1$, $C_1=P$ (by Remark 1), and one chooses $D_1=C_1$.

Assume inductively that the assertion is true for $k-1$, $k \geq 2$, and let P, C_1, \dots, C_k be given as described in the statement of Theorem 2. Since the inductive assumption takes care of the cases in which $C_i = \emptyset$ for some $1 \leq i \leq k$, we assume that $C_i \neq \emptyset$ for all $1 \leq i \leq k$. If $C_i \cap C_j = \emptyset$ for all $1 \leq i < j \leq k$, then the existence of D_1 with the required property is guaranteed by Theorem 3, which is assumed to hold. Otherwise, let m and n be such that $C_m \cap C_n \neq \emptyset$ and $1 \leq m < n \leq k$, and let $x \in C_m \cap C_n$. Define C_1^*, \dots, C_{k-1}^* by

$$C_i^* = \begin{cases} \text{conv}(C_m \cup C_n) & \text{if } i = m, \\ C_i & \text{if } i < m \text{ or } m < i < n, \\ C_{i+1} & \text{if } i \geq n. \end{cases}$$

Clearly, P together with C_1^*, \dots, C_{k-1}^* satisfy all the conditions of Theorem 2, hence the inductive assumption implies that there exists a polytope D_1^* in C_1^* such that every hyperplane that meets P meets $D_1^* \cup \bigcup_{i=2}^k C_i^*$.

Case 1. $m \neq 1$. Choose $D_1 = D_1^*$. If H is a hyperplane that meets P , then H meets $D_1 \cup \bigcup_{i=2}^{k-1} C_i^*$, i.e. H meets $D_1 \cup \bigcup_{i \geq 2, i \neq m} C_i \cup \text{conv}(C_m \cup C_n)$. If H meets $D_1 \cup \bigcup_{i \geq 2, i \neq m} C_i$, then clearly H meets $D_1 \cup \bigcup_{i=2}^k C_i$, as required. If H meets $\text{conv}(C_m \cup C_n)$, then by Lemma 1, H meets $\text{conv}(C_m \cup y) \cup \text{conv}(C_m \cup x)$ for every $y \in E^d$, hence in particular H meets $\text{conv}(C_m \cup x) \cup \text{conv}(C_n \cup x)$ (where $x \in C_m \cap C_n$); this means that H meets $C_m \cup C_n$, because C_m and C_n are convex sets. Therefore H meets $D_1 \cup \bigcup_{i=2}^k C_i$, as required.

Case 2. $m=1$. In this case $D_1^* \subset C_1^* = \text{conv}(C_1 \cup C_n)$. Every vertex of D_1^* is a finite convex combination of points of C_1 and points of C_n , therefore there exists a polytope D_1^{**} in $\text{conv}(C_1 \cup C_n)$ such that $D_1^{**} \supset D_1^*$ and $\text{Vert}(D_1^{**}) \subset C_1 \cup C_n$.

Define D_1 by $D_1 = \text{conv}[x \cup (\text{Vert } D_1^{**} \cap C_1)]$, where $x \in C_1 \cap C_n$. Let H be an arbitrary hyperplane that meets P ; H meets $D_1^* \cup \bigcup_{i=2}^{k-1} C_i^* = D_1^* \cup \bigcup_{i \geq 2, i \neq n} C_i$, and since $D_1^{**} \supset D_1^*$, H meets $D_1^{**} \cup \bigcup_{i \geq 2, i \neq n} C_i$. If H meets $\bigcup_{i \geq 2, i \neq n} C_i$, then clearly H meets $D_1 \cup \bigcup_{i=2}^k C_i$, as required. If H meets D_1^{**} , then by Corollary 1, H meets $\text{conv}[x \cup (\text{Vert } D_1^{**} \cap C_1)] \cup \text{conv}[x \cup (\text{Vert } D_1^{**} \cap C_n)]$; the first set in this union is D_1 , by the definition

of D_1 , and therefore if H meets D_1 it clearly meets $D_1 \cup \bigcup_{i=2}^k C_i$, as required. If, however, H meets $\text{conv}[x \cup \text{Vert}(D_1^{**} \cap C_n)]$, then H meets C_n , since $x \in C_1 \cap C_n \subset C_n$ and C_n is a convex set; therefore H meets $D_1 \cup \bigcup_{i=2}^k C_i$, and Claim 1 has been established.

PROOF OF THEOREM 3. By induction on k , starting with the case $k=1$ being true by Remark 1. Assume inductively that the assertion is true for $k-1$, $k \geq 2$, and let P, C_1, \dots, C_k be given as described in the statement of Theorem 3 (assume as before that $C_i \neq \emptyset$ for all $1 \leq i \leq k$).

Suppose first that for some j , $1 \leq j \leq k$, every hyperplane that meets P meets $\bigcup_{i \neq j} C_i$. If $j \neq 1$, then by the inductive assumption applied to $P, C_1, \dots, C_{j-1}, C_{j+1}, \dots, C_k$, there exists a polytope D_1 in C_1 , such that every hyperplane that meets P meets $D_1 \cup (\bigcup_{i \neq 1, j} C_i)$; therefore every hyperplane that meets P meets also $D_1 \cup \bigcup_{i=2}^k C_i$, as required. If $j=1$, then any choice for a D_1 in C_1 will do.

If however there is no such a j , then there are hyperplanes that meet C_1 and do not meet $\bigcup_{i=2}^k C_i$.

For every hyperplane H with $H \cap C_1 \neq \emptyset$ and $H \cap (\bigcup_{i \neq 1} C_i) = \emptyset$ define K_1 and K_2 by

$$K_1 = \text{conv} \bigcup \{C_i \mid C_i \subset \mathring{H}_+\} \quad \text{and} \quad K_2 = \text{conv} \bigcup \{C_i \mid C_i \subset \mathring{H}_-\}.$$

Clearly $K_1 \subset \mathring{H}_+$ and $K_2 \subset \mathring{H}_-$, hence $K_1 \cap K_2 = \emptyset$; not both of K_1 and K_2 are empty since $H \cap (\bigcup_{i \neq 1} C_i) = \emptyset$, $k \geq 2$ and $C_i \neq \emptyset$ for all $1 \leq i \leq k$.

Claim 2. If $K_1 \neq \emptyset$ and $K_2 \neq \emptyset$, then C contains a segment L such that if a hyperplane F separates K_1 and K_2 then $F \cap L \neq \emptyset$.

PROOF. If $K_1 \cap \text{conv}(K_2 \cup C_1) = \emptyset$, then K_1 and $\text{conv}(K_2 \cup C_1)$ can be strictly separated by a hyperplane F_0 ; hence $F_0 \cap \bigcup_{i=1}^k C_i = \emptyset$; however $K_1 \neq \emptyset$, $\text{conv}(K_2 \cup C_1) \neq \emptyset$ and P being convex imply that $F_0 \cap P \neq \emptyset$, which contradicts the assumption on P, C_1, \dots, C_k . Therefore $K_1 \cap \text{conv}(K_2 \cup C_1) \neq \emptyset$ and similarly $K_2 \cap \text{conv}(K_1 \cup C_1) \neq \emptyset$.

Take $x_1 \in K_1 \cap \text{conv}(K_2 \cup C_1)$; then there are points $y_1 \in K_2$ and $z_1 \in C_1$ such that $x_1 = \lambda y_1 + (1-\lambda)z_1$, for some $0 \leq \lambda \leq 1$; similarly take $y_2 \in K_2 \cap \text{conv}(K_1 \cup C_1)$, then there are points $x_2 \in K_1$ and $z_2 \in C_1$ such that $y_2 = \mu x_2 + (1-\mu)z_2$, for some $0 \leq \mu \leq 1$ (see Figure 1).

The promised segment L in C_1 is taken as the segment $[z_1 z_2]$. Suppose a hyperplane F separates K_1 and K_2 , so that say $K_1 \subset F_+$ and $K_2 \subset F_-$. Clearly $x_1, x_2 \in K_1 \subset \mathring{F}_+$ and $y_1, y_2 \in K_2 \subset \mathring{F}_-$. Since $x_1 = \lambda y_1 + (1-\lambda)z_1$ and $0 \leq \lambda \leq 1$ it follows that $z_1 \in F_+$, and similarly $y_2 = \mu x_2 + (1-\mu)z_2$ and $0 \leq \mu \leq 1$ imply $z_2 \in F_-$. As a result F meets the segment $[z_1 z_2] = L$, and Claim 2 has been established.

Claim 3. If $K_1 = \emptyset$, then C_1 contains a segment L such that if a hyperplane F meets C_1 and $\bigcup_{i \neq 1} C_i \subset \mathring{F}_+$ (or $\bigcup_{i \neq 1} C_i \subset \mathring{F}_-$), then

$$F \cap \text{conv}[L \cup (\text{Ext } P \cap C_1)] \neq \emptyset.$$

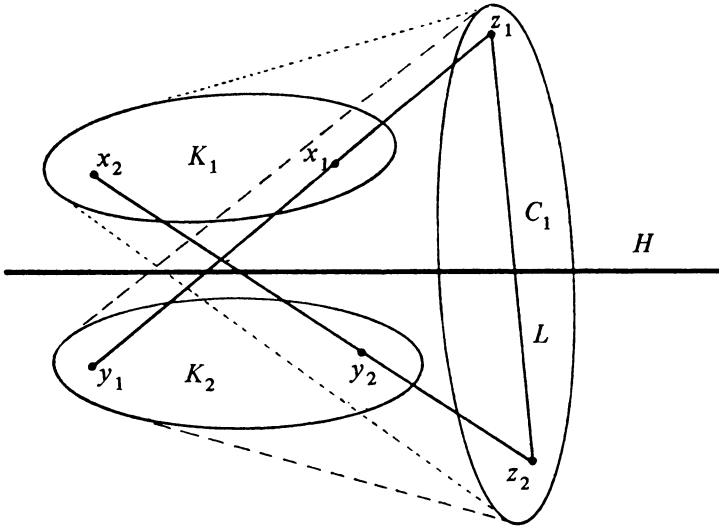


FIGURE 1

PROOF. $C_1 \cap \text{conv}(\bigcup_{i \neq 1} C_i) \neq \emptyset$ since otherwise C_1 and $\text{conv}(\bigcup_{i \neq 1} C_i)$ are strictly separated by a hyperplane F_0 , hence $F_0 \cap (\bigcup_{i=1}^k C_i) = \emptyset$; since P is convex it follows that $F_0 \cap P \neq \emptyset$ which contradicts the assumption on P, C_1, \dots, C_k . Let $x \in C_1 \cap \text{conv}(\bigcup_{i \neq 1} C_i)$, and take for the segment L any segment in C_1 containing x (in fact $L = \{x\}$ is as good).

Suppose F is a hyperplane that meets C_1 and $\bigcup_{i \neq 1} C_i \subset \hat{F}_+$; therefore $\text{conv}(\bigcup_{i \neq 1} C_i) \subset \hat{F}_+$ and hence $x \in \hat{F}_+$. Next $F \cap C_1 \neq \emptyset$, hence $F_- \cap C_1 \neq \emptyset$ and therefore $F_- \cap P \neq \emptyset$; hence $F_- \cap \text{Ext } P \neq \emptyset$. Moreover $F_- \cap \text{Ext } P \subset C_1$, because $\text{Vert } P \subset \bigcup_{i=1}^k C_i$ (see Remark 1), and hence $F_- \cap \text{Ext } P \subset F_- \cap \bigcup_{i=1}^k C_i = F_- \cap C_1 \subset C_1$.

We conclude that $x \in \hat{F}_+$ and C_1 contains a vertex y of P with $y \in F_-$; therefore $F \cap \text{conv}[L \cup (\text{Ext } P \cap C_1)] \neq \emptyset$.

Claim 3 has been established.

Let L_1, \dots, L_r be a collection of segments in C_1 , each one obtained by applying Claims 2 and 3 to each and every different division $\{2, \dots, k\} = I \cup J$ with $I \cap J = \emptyset$, for which there exists a hyperplane H with $H \cap C_1 \neq \emptyset, \hat{H}_+ \supset \bigcup_{i \in I} C_i$ and $\hat{H}_- \supset \bigcup_{i \in J} C_i$.

Define D_1 by $D_1 = \text{conv}\{\bigcup_{i=1}^r L_i \cup (\text{Ext } P \cap C_1)\}$. To show that D_1 has the required property as claimed in Theorem 3, suppose a hyperplane H meets P . If $P \cap (\bigcup_{i \neq 1} C_i) \neq \emptyset$, then clearly $P \cap (D_1 \cup \bigcup_{i \neq 1} C_i) \neq \emptyset$. Otherwise $P \cap (\bigcup_{i \neq 1} C_i) = \emptyset$, and since $H \cap P \neq \emptyset$, it follows that $H \cap C_1 \neq \emptyset$. By Claim 2 or 3, C_1 contains the segment L_j for the appropriate $j, 1 \leq j \leq r$, such that $H \cap \text{conv}[L_j \cup (\text{Ext } P \cap C_1)] \neq \emptyset$, hence $H \cap D_1 \neq \emptyset$ as needed. D_1 is clearly a polytope in C_1 .

This completes the proof of Theorem 3.

The proof of Theorem 1 follows now easily from the proof of Theorem 3, which implies Theorem 2 by Claim 1, and the equivalence of Theorems 2 and 1.

REMARK 2. Shortly before proving conjecture $(d, d-1, k)$ for all $d \geq 2$ and $k \geq 1$, we established conjecture $(2, 1, k)$ for all $k \geq 1$, using the following:

Claim 4. If C_1 and C_2 are disjoint compact convex sets in E^2 , then they have at most four (4) common supporting lines.

Claim 5. If C_1 and C_2 are disjoint compact convex sets in E^d , $d \geq 2$, $\{H_i | i \in I\}$ the collection of all the common supporting hyperplanes to C_1 and C_2 , $x_i \in H_i \cap C_1$ and $y_i \in H_i \cap C_2$ for all $i \in I$, then every hyperplane that meets both C_1 and C_2 meets $\text{conv}\{x_i | i \in I\} \cup \text{conv}\{y_i | i \in I\}$.

Both Claims 4 and 5 in the case $d=2$ imply the following: "If C_1 and C_2 are disjoint compact convex sets in E^2 , then there exist convex quadrangles D_1 and D_2 , $D_i \subseteq C_i$ for $i=1, 2$, such that every hyperplane that meets both C_1 and C_2 meets $D_1 \cup D_2$ ". Unfortunately, the index set I in Claim 5 is infinite for all $d \geq 3$, and there is no valid analogue of the last theorem for E^d , $d \geq 3$, with "convex quadrangle" replaced by "polytopes" (take, for example, two disjoint balls).

REMARK 3. Lemma 1 can be extended as follows:

LEMMA 2. If C is a convex set in E^d , $x \in E^d$, and $\{A_i | i \in I\}$ are such that $C = \text{conv} \bigcup \{A_i | i \in I\}$, then every hyperplane that meets C meets $\bigcup_{i \in I} \text{conv}(x \cup A_i)$.

The proof is similar to the proof of Lemma 1, hence it is omitted.

COROLLARY 2. If P is a polytope in E^d , $x \in E^d$, and $\{v_1, \dots, v_n\} = \text{Vert } P$, then every hyperplane that meets P meets $\bigcup_{i=1}^n [x, v_i]$.

Let a graph (= finite 1-dimensional simplicial complex) be called *starshape* if it has exactly $n+1$ vertices, one of valence n and n of valence 1, $n \geq 1$.

COROLLARY 3. If P is a d -polytope in E^d , C_1, \dots, C_k are closed convex subsets of P , such that every hyperplane that meets P meets $\bigcup_{i=1}^k C_i$, then there exist starshapes G_1, \dots, G_k with $G_i \subseteq C_i$ for all $1 \leq i \leq k$, such that every hyperplane that meets P meets $\bigcup_{i=1}^k G_i$.

PROOF. There exist, by Theorem 1, polytopes D_1, \dots, D_k with $D_i \subseteq C_i$ for all $1 \leq i \leq k$, such that every hyperplane that meets P meets $\bigcup_{i=1}^k D_i$; let $x_i \in D_i$, and define G_i by $G_i = \bigcup \{[x, v_i] | v_i \in \text{Vert } D_i\}$, for all $1 \leq i \leq k$. G_i is a starshape, for all $1 \leq i \leq k$, and every hyperplane that meets $\bigcup_{i=1}^k D_i$ meets $\bigcup_{i=1}^k G_i$, by Corollary 2.

REMARK 4. Conjecture (3, 1, 3) has been recently established by the author of this paper, using some ideas of [2]; the proof will appear.

REMARK 5. As stated in [3], it was M. O. Rabin who first proposed conjecture $(d, 0, k)$, for all $d \geq 1$ and $k \geq 1$.

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