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ON A CONJECTURE OF A. J. HOFFMAN. II

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ABSTRACT. It is proved that certain incidence relations of hyperplanes and closed convex sets in a d-polytope can be preserved while replacing these sets by suitable polytopal subsets.

The purpose of this paper is to prove

THEOREM 1. *If P is a d-polytope in E^d and $C_1, \cdots, C_k$ are closed convex subsets of P, such that every hyperplane that meets P meets $\bigcup_{i=1}^k C_i$, then there exist polytopes $D_1, \cdots, D_k$ with $D_i \subseteq C_i$ for all $1 \leqq i \leqq k$, such that every hyperplane that meets P meets $\bigcup_{i=1}^k D_i$.*

This settles all the cases $(d, d-1, k)$, for all $d \geqq 2$ and $k \geqq 1$, of the following conjecture due to A. J. Hoffman [3]:

Conjecture (d, t, k). If P is a d-polytope in E^d, $d \geqq 1$; $t \geqq 0$ and $k \geqq 1$ are integers, $C_1, \cdots, C_k$ are closed convex subsets of P such that every (affine) t-flat that meets P meets $\bigcup_{i=1}^k C_i$; then there are polytopes $D_1, \cdots, D_k$ with $D_i \subseteq C_i$ for all $1 \leqq i \leqq k$, such that every t-flat that meets P meets $\bigcup_{i=1}^k D_i$.

A. J. Hoffman proved [3] conjecture $(d, 0, k)$, for all $d \geqq 1$ and $k \geqq 1$; in these cases the t-flats are points and $C_1, \cdots, C_k$ cover P.

It follows quite elementarily that conjecture $(d, t, 1)$ is true for all $d \geqq 1$ and $t \geqq 0$, since in this case $C_1 = P$ (see Remark 1, here). W. R. Hare, Jr. and C. R. Smith proved [2] that conjecture $(d, t, 2)$ is true for all $d \geqq 1$ and $t \geqq 0$. We have previously shown [4] that conjecture $(d, d-2, k)$ is false for all $d \geqq 3$ and $k \geqq 4$, while here it is shown that conjecture $(d, d-1, k)$ is true for all $d \geqq 2$ and $k \geqq 1$. Conjecture $(3, 1, 3)$ is true (see Remark 4).

DEFINITIONS. A *polytope* P is the convex hull of a finite set of points in the Euclidean d-dimensional space E^d; a *d-polytope* in E^d is a polytope with nonempty interior; Vert P denotes here the set of vertices of a polytope P; if $A \subset E^d$, conv A denotes the convex hull of A.

An (affine) *t-flat* in E^d is a translate of a t-dimensional subspace of E^d, and a *hyperplane* is a $(d-1)$-flat. If H is a hyperplane in E^d, then H_+ and H_- ($\mathring{H}_+$ and $\mathring{H}_-$) denote the two closed (open, respectively) half-spaces of E^d, determined by H.

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A hyperplane H *supports* a compact set X in E^d if $X \subset H_+$ and $H \cap X \neq \varnothing$. A hyperplane H *separates* (*strictly separates*) A and B if $A \subset H_+$ and $B \subset H_-$ ($A \subset \mathring{H}_+$ and $B \subset \mathring{H}_-$, respectively). For additional definitions and information the reader is referred to **[1]**.

REMARK 1. If $P, C_1, \cdots, C_k$ are as given in conjecture (d, t, k), for some $d \geqq 1$, $k \geqq 1$ and $0 \leqq t \leqq d-1$, then $\operatorname{Vert} P \subseteq \bigcup_{i=1}^k C_i$. To establish this, let $v \in \operatorname{Vert} P$; there exists a hyperplane H_v of P such that $H_v \cap P = \{v\}$; H_v clearly contains a t-flat F_v through v. Since F_v is a t-flat that meets P (at v), it follows by the assumptions that F_v meets $\bigcup_{i=1}^k C_i$, i.e. $F_v \cap (\bigcup_{i=1}^k C_i) \neq \varnothing$, and since $\bigcup_{i=1}^k C_i \subset P$, $F_v \cap (\bigcup_{i=1}^k C_i) \subset F_v \cap P = \{v\}$; hence $v \in \bigcup_{i=1}^k C_i$ and therefore $\operatorname{Vert} P \subseteq \bigcup_{i=1}^k C_i$.

In case $k=1$, it follows that $\operatorname{Vert} P \subseteq C_1$, and therefore $C_1 = P$, since C_1 is a convex set contained in P. As a result, conjecture $(d, t, 1)$ is (trivially) true for all $d \geqq 1$ and $0 \leqq t \leqq d-1$.

We need the following:

LEMMA 1. *If C is a convex set in E^d, $x \in E^d$, and A and B are such that $C = \operatorname{conv}(A \cup B)$, then every hyperplane that meets C meets* $\operatorname{conv}(A \cup x) \cup \operatorname{conv}(B \cup x)$.

PROOF. Let Y be defined by $Y = \operatorname{conv}(A \cup x) \cup \operatorname{conv}(B \cup x)$; every pair of points y_1 and y_2 of Y are connected by the polyhedral path $y_1 x \cup x y_2$ which lies entirely in Y.

Let H be an arbitrary hyperplane such that $H \cap C \neq \varnothing$. If $A \subset \mathring{H}_+$ and $B \subset \mathring{H}_+$, then $\operatorname{conv}(A \cup B) \subset \mathring{H}_+$, hence $C \subset \mathring{H}_+$, but this implies $C \cap H = \varnothing$—a contradiction; therefore $A \cup B \not\subset \mathring{H}_+$, and similarly $A \cup B \not\subset \mathring{H}_-$. Therefore $A \cup B$ contains a point y_1 in H_+ and a point y_2 in H_-; $y_1, y_2 \in Y$ since $Y \supset A \cup B$. The polyhedral path in Y that connects y_1 to y_2 clearly meets H, hence $Y \cap H \neq \varnothing$ and the proof is complete.

COROLLARY 1. *If P is a polytope in E^d, $x \in E^d$, and $\operatorname{Vert} P = A \cup B$, then every hyperplane that meets P meets* $\operatorname{conv}(A \cup x) \cup \operatorname{conv}(B \cup x)$.

PROOF. In this case $P = \operatorname{conv}(\operatorname{Vert} P) = \operatorname{conv}(A \cup B)$, and Lemma 1 is applicable (with $P = C$).

Since the replacement of k compact convex sets by polytopes can be done one at a time, we state and prove the following:

THEOREM 2. *If P is a d-polytope in E^d, $C_1, \cdots, C_k$ are closed convex subsets of P such that every hyperplane that meets P meets $\bigcup_{i=1}^k C_i$, then there exists a polytope D_1 in C_1 such that every hyperplane that meets P meets $D_1 \cup \bigcup_{i=2}^k C_i$.*

THEOREM 3. *If P is a d-polytope in E^d, $C_1, \cdots, C_k$ are closed convex subsets of P, $C_i \cap C_j = \varnothing$ for all $1 \leqq i < j \leqq k$ and every hyperplane that meets*

P meets $\bigcup_{i=1}^{k} C_i$, then there exists a polytope D_1 in C_1 such that every hyperplane that meets P meets $D_1 \cup \bigcup_{i=2}^{k} C_i$.

Clearly Theorems 1 and 2 are equivalent; our proof of Theorem 1 uses the following.

Claim 1. Theorem 3 implies Theorem 2.

PROOF OF CLAIM 1. Assuming Theorem 3 is true, we prove Theorem 2 by induction on k. In Case $k=1$, $C_1=P$ (by Remark 1), and one chooses $D_1=C_1$.

Assume inductively that the assertion is true for $k-1$, $k\geqq 2$, and let $P, C_1, \cdots, C_k$ be given as described in the statement of Theorem 2. Since the inductive assumption takes care of the cases in which $C_i=\varnothing$ for some $1\leqq i\leqq k$, we assume that $C_i\neq\varnothing$ for all $1\leqq i\leqq k$. If $C_i\cap C_j=\varnothing$ for all $1\leqq i<j\leqq k$, then the existence of D_1 with the required property is guaranteed by Theorem 3, which is assumed to hold. Otherwise, let m and n be such that $C_m\cap C_n\neq\varnothing$ and $1\leqq m<n\leqq k$, and let $x\in C_m\cap C_n$. Define $C_1^*, \cdots, C_{k-1}^*$ by

$$C_i^* = \begin{cases} \operatorname{conv}(C_m \cup C_n) & \text{if } i = m, \\ C_i & \text{if } i < m \text{ or } m < i < n, \\ C_{i+1} & \text{if } i \geqq n. \end{cases}$$

Clearly, P together with $C_1^*, \cdots, C_{k-1}^*$ satisfy all the conditions of Theorem 2, hence the inductive assumption implies that there exists a polytope D_1^* in C_1^* such that every hyperplane that meets P meets $D_1^*\cup\bigcup_{i=2}^{k} C_i^*$.

Case 1. $m\neq 1$. Choose $D_1=D_1^*$. If H is a hyperplane that meets P, then H meets $D_1\cup\bigcup_{i=2}^{k-1} C_i^*$, i.e. H meets $D_1\cup\bigcup_{i\geqq 2; i\neq m} C_i\cup\operatorname{conv}(C_m\cup C_n)$. If H meets $D_1\cup\bigcup_{i\geqq 2; i\neq m} C_i$, then clearly H meets $D_1\cup\bigcup_{i=2}^{k} C_i$, as required. If H meets $\operatorname{conv}(C_m\cup C_n)$, then by Lemma 1, H meets $\operatorname{conv}(C_m\cup y)\cup\operatorname{conv}(C_m\cup y)$ for every $y\in E^d$, hence in particular H meets $\operatorname{conv}(C_m\cup x)\cup\operatorname{conv}(C_n\cup x)$ (where $x\in C_m\cap C_n$); this means that H meets $C_m\cup C_n$, because C_m and C_n are convex sets. Therefore H meets $D_1\cup\bigcup_{i=2}^{k} C_i$, as required.

Case 2. $m=1$. In this case $D_1^*\subset C_1^*=\operatorname{conv}(C_1\cup C_n)$. Every vertex of D_1^* is a finite convex combination of points of C_1 and points of C_2, therefore there exists a polytope D_1^{**} in $\operatorname{conv}(C_1\cup C_n)$ such that $D_1^{**}\supset D_1^*$ and $\operatorname{Vert}(D_1^{**})\subset C_1\cup C_n$.

Define D_1 by $D_1=\operatorname{conv}[x\cup(\operatorname{Vert} D_1^{**}\cap C_1)]$, where $x\in C_1\cap C_n$. Let H be an arbitrary hyperplane that meets P; H meets $D_1^*\cup\bigcup_{i=2}^{k-1} C_i=$ $D_1^*\cup\bigcup_{i\geqq 2; i\neq n} C_i$, and since $D_1^{**}\supset D_1^*$, H meets $D_1^{**}\cup\bigcup_{i\geqq 2; i\neq n} C_i$. If H meets $\bigcup_{i\geqq 2; i\neq n} C_i$, then clearly H meets $D_1\cup\bigcup_{i=2}^{k} C_i$, as required. If H meets D_1^{**}, then by Corollary 1, H meets $\operatorname{conv}[x\cup(\operatorname{Vert} D_1^{**}\cap C_1)]\cup$ $\operatorname{conv}[x\cup(\operatorname{Vert} D_1^{**}\cap C_n)]$; the first set in this union is D_1, by the definition of D_1, and therefore if H meets D_1 it clearly meets $D_1 \cup \bigcup_{i=2}^k C_i$, as required. If, however, H meets $\operatorname{conv}[x \cup \operatorname{Vert}(D_1^{**} \cap C_n)]$, then H meets C_n, since $x \in C_1 \cap C_n \subset C_n$ and C_n is a convex set; therefore H meets $D_1 \cup \bigcup_{i=2}^k C_i$, and Claim 1 has been established.

PROOF OF THEOREM 3. By induction on k, starting with the case $k=1$ being true by Remark 1. Assume inductively that the assertion is true for $k-1$, $k \geqq 2$, and let $P, C_1, \cdots, C_k$ be given as described in the statement of Theorem 3 (assume as before that $C_i \neq \varnothing$ for all $1 \leqq i \leqq k$).

Suppose first that for some j, $1 \leqq j \leqq k$, every hyperplane that meets P meets $\bigcup_{i \neq j} C_i$. If $j \neq 1$, then by the inductive assumption applied to P, $C_1, \cdots, C_{j-1}, C_{j+1}, \cdots, C_k$, there exists a polytope D_1 in C_1, such that every hyperplane that meets P meets $D_1 \cup (\bigcup_{i \neq 1, j} C_i)$; therefore every hyperplane that meets P meets also $D_1 \cup \bigcup_{i=2}^k C_i$, as required. If $j=1$, then any choice for a D_1 in C_1 will do.

If however there is no such a j, then there are hyperplanes that meet C_1 and do not meet $\bigcup_{i=2}^k C_i$.

For every hyperplane H with $H \cap C_1 \neq \varnothing$ and $H \cap (\bigcup_{i \neq 1} C_i) = \varnothing$ define K_1 and K_2 by

$$K_1 = \operatorname{conv} \bigcup \{C_i \mid C_i \subset \mathring{H}_+\} \quad \text{and} \quad K_2 = \operatorname{conv} \bigcup \{C \mid C_i \subset \mathring{H}_-\}.$$

Clearly $K_1 \subset \mathring{H}_+$ and $K_2 \subset \mathring{H}_-$, hence $K_1 \cap K_2 = \varnothing$; not both of K_1 and K_2 are empty since $H \cap (\bigcup_{i \neq 1} C_i) = \varnothing$, $k \geqq 2$ and $C_i \neq \varnothing$ for all $1 \leqq i \leqq k$.

Claim 2. If $K_1 \neq \varnothing$ and $K_2 \neq \varnothing$, then C contains a segment L such that if a hyperplane F separates K_1 and K_2 then $F \cap L \neq \varnothing$.

PROOF. If $K_1 \cap \operatorname{conv}(K_2 \cup C_1) = \varnothing$, then K_1 and $\operatorname{conv}(K_2 \cup C_1)$ can be strictly separated by a hyperplane F_0; hence $F_0 \cap \bigcup_{i=1}^k C_i = \varnothing$; however $K_1 \neq \varnothing$, $\operatorname{conv}(K_2 \cup C_1) \neq \varnothing$ and P being convex imply that $F_0 \cap P \neq \varnothing$, which contradicts the assumption on $P, C_1, \cdots, C_k$. Therefore $K_1 \cap \operatorname{conv}(K_2 \cup C_1) \neq \varnothing$ and similarly $K_2 \cap \operatorname{conv}(K_1 \cup C_1) \neq \varnothing$.

Take $x_1 \in K_1 \cap \operatorname{conv}(K_2 \cup C_1)$; then there are points $y_1 \in K_2$ and $z_1 \in C_1$ such that $x_1 = \lambda y_1 + (1-\lambda) z_1$, for some $0 \leqq \lambda \leqq 1$; similarly take $y_2 \in K_2 \cap \operatorname{conv}(K_1 \cup C_1)$, then there are points $x_2 \in K_1$ and $z_2 \in C_1$ such that $y_1 = \mu x_2 + (1-\mu) z_2$, for some $0 \leqq \mu \leqq 1$ (see Figure 1).

The promised segment L in C_1 is taken as the segment $[z_1 z_2]$. Suppose a hyperplane F separates K_1 and K_2, so that say $K_1 \subset F_+$ and $K_2 \subset F_-$. Clearly $x_1, x_2 \in K_1 \subset \mathring{F}_+$ and $y_1, y_2 \in K_2 \subset \mathring{F}_-$. Since $x_1 = \lambda y_1 + (1-\lambda) z_1$ and $0 \leqq \lambda \leqq 1$ it follows that $z_1 \in F_+$, and similarly $y_2 = \mu x_2 + (1-\mu) z_2$ and $0 \leqq \mu \leqq 1$ imply $z_2 \in F_-$. As a result F meets the segment $[z_1 z_2] = L$, and Claim 2 has been established.

Claim 3. If $K_1 = \varnothing$, then C_1 contains a segment L such that if a hyperplane F meets C_1 and $\bigcup_{i \neq 1} C_i \subset \mathring{F}_+$ (or $\bigcup_{i \neq 1} C_i \subset \mathring{F}_-$), then

$$F \cap \operatorname{conv}[L \cup (\operatorname{Ext} P \cap C_1)] \neq \varnothing.$$

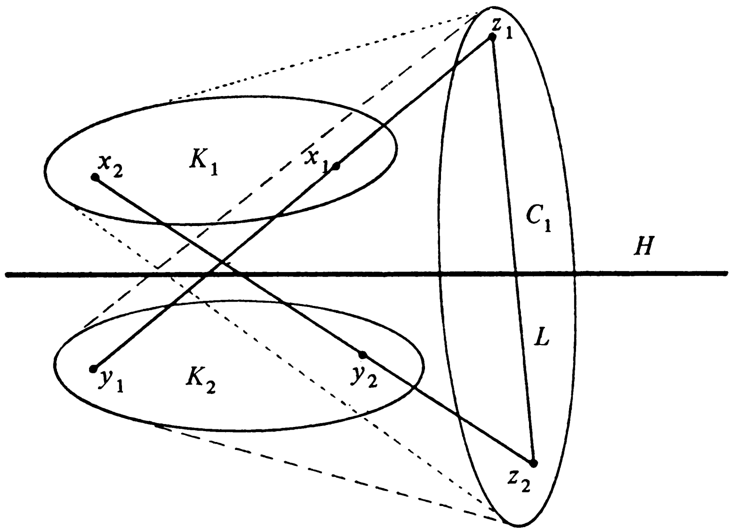

FIGURE 1

PROOF. $C_1 \cap \operatorname{conv}(\bigcup_{i\neq 1} C_i) \neq \varnothing$ since otherwise C_1 and $\operatorname{conv}(\bigcup_{i\neq 1} C_i)$ are strictly separated by a hyperplane F_0, hence $F_0 \cap (\bigcup_{i=1}^k C_i) = \varnothing$; since P is convex it follows that $F_0 \cap P \neq \varnothing$ which contradicts the assumption on $P, C_1, \cdots, C_k$. Let $x \in C_1 \cap \operatorname{conv}(\bigcup_{i\neq 1} C_i)$, and take for the segment L any segment in C_1 containing x (in fact $L=\{x\}$ is as good).

Suppose F is a hyperplane that meets C_1 and $\bigcup_{i\neq 1} C_i \subset \mathring{F}_+$; therefore $\operatorname{conv}(\bigcup_{i\neq 1} C_i) \subset \mathring{F}_+$ and hence $x \in \mathring{F}_+$. Next $F \cap C_1 \neq \varnothing$, hence $F_- \cap C_1 \neq \varnothing$ and therefore $F_- \cap P \neq \varnothing$; hence $F_- \cap \operatorname{Ext} P \neq \varnothing$. Moreover $F_- \cap \operatorname{Ext} P \subset C_1$, because $\operatorname{Vert} P \subset \bigcup_{i=1}^k C_i$ (see Remark 1), and hence $F_- \cap \operatorname{Ext} P \subset F_- \cap \bigcup_{i=1}^k C_i = F_- \cap C_1 \subset C_1$.

We conclude that $x \in \mathring{F}_+$ and C_1 contains a vertex y of P with $y \in F_-$; therefore $F \cap \operatorname{conv}[L \cup (\operatorname{Ext} P \cap C_1)] \neq \varnothing$.

Claim 3 has been established.

Let $L_1, \cdots, L_r$ be a collection of segments in C_1, each one obtained by applying Claims 2 and 3 to each and every different division $\{2, \cdots, k\} = I \cup J$ with $I \cap J = \varnothing$, for which there exists a hyperplane H with $H \cap C_1 \neq \varnothing$, $\mathring{H}_+ \supset \bigcup_{i\in I} C_i$ and $\mathring{H}_- \supset \bigcup_{i\in J} C_i$.

Define D_1 by $D_1 = \operatorname{conv}\{\bigcup_{i=1}^r L_i \cup (\operatorname{Ext} P \cap C_1)\}$. To show that D_1 has the required property as claimed in Theorem 3, suppose a hyperplane H meets P. If $P \cap (\bigcup_{i\neq 1} C_i) \neq \varnothing$, then clearly $P \cap (D_1 \cup \bigcup_{i\neq 1} C_i) \neq \varnothing$. Otherwise $P \cap (\bigcup_{i\neq 1} C_i) = \varnothing$, and since $H \cap P \neq \varnothing$, it follows that $H \cap C_1 \neq \varnothing$. By Claim 2 or 3, C_1 contains the segment L_j for the appropriate j, $1 \leqq j \leqq r$, such that $H \cap \operatorname{conv}[L_j \cup (\operatorname{Ext} P \cap C_1)] \neq \varnothing$, hence $H \cap D_1 \neq \varnothing$ as needed. D_1 is clearly a polytope in C_1.

This completes the proof of Theorem 3.

The proof of Theorem 1 follows now easily from the proof of Theorem 3, which implies Theorem 2 by Claim 1, and the equivalence of Theorems 2 and 1.

REMARK 2. Shortly before proving conjecture $(d, d-1, k)$ for all $d \geqq 2$ and $k \geqq 1$, we established conjecture $(2, 1, k)$ for all $k \geqq 1$, using the following:

Claim 4. If C_1 and C_2 are disjoint compact convex sets in E^2, then they have at most four (4) common supporting lines.

Claim 5. If C_1 and C_2 are disjoint compact convex sets in E^d, $d \geqq 2$, $\{H_i | i \in I\}$ the collection of all the common supporting hyperplanes to C_1 and C_2, $x_i \in H_i \cap C_1$ and $y_i \in H_i \cap C_2$ for all $i \in I$, then every hyperplane that meets *both* C_1 and C_2 meets $\text{conv}\{x_i | i \in I\} \cup \text{conv}\{y_i | i \in I\}$.

Both Claims 4 and 5 in the case $d=2$ imply the following: "If C_1 and C_2 are disjoint compact convex sets in E^2, then there exist convex quadrangles D_1 and D_2, $D_i \subseteq C_i$ for $i=1, 2$, such that every hyperplane that meets both C_1 and C_2 meets $D_1 \cup D_2$". Unfortunately, the index set I in Claim 5 is infinite for all $d \geqq 3$, and there is no valid analogue of the last theorem for E^d, $d \geqq 3$, with "convex quadrangle" replaced by "polytopes" (take, for example, two disjoint balls).

REMARK 3. Lemma 1 can be extended as follows:

LEMMA 2. *If C is a convex set in E^d, $x \in E^d$, and $\{A_i | i \in I\}$ are such that $C = \text{conv} \bigcup \{A_i | i \in I\}$, then every hyperplane that meets C meets $\bigcup_{i \in I} \text{conv}(x \cup A_i)$.*

The proof is similar to the proof of Lemma 1, hence it is omitted.

COROLLARY 2. *If P is a polytope in E^d, $x \in E^d$, and $\{v_1, \cdots . v_n\} = \text{Vert}\, P$, then every hyperplane that meets P meets $\bigcup_{i=1}^{n} [x, v_i]$.*

Let a graph (= finite 1-dimensional simplicial complex) be called *starshape* if it has exactly $n+1$ vertices, one of valence n and n of valence 1, $n \geqq 1$.

COROLLARY 3. *If P is a d-polytope in E^d, $C_1, \cdots, C_k$ are closed convex subsets of P, such that every hyperplane that meets P meets $\bigcup_{i=1}^{k} C_i$, then there exist starshapes $G_1, \cdots, G_k$ with $G_i \subseteq C_i$ for all $1 \leqq i \leqq k$, such that every hyperplane that meets P meets $\bigcup_{i=1}^{k} G_i$.*

PROOF. There exist, by Theorem 1, polytopes $D_1, \cdots, D_k$ with $D_i \subseteq C_i$ for all $1 \leqq i \leqq k$, such that every hyperplane that meets P meets $\bigcup_{i=1}^{k} D_i$; let $x_i \in D_i$, and define G_i by $G_i = \bigcup \{[x, v_i] | v_i \in \text{Vert}\, D_i\}$, for all $1 \leqq i \leqq k$. G_i is a starshape, for all $1 \leqq i \leqq k$, and every hyperplane that meets $\bigcup_{i=1}^{k} D_i$ meets $\bigcup_{i=1}^{k} G_i$, by Corollary 2.

REMARK 4. Conjecture (3, 1, 3) has been recently established by the author of this paper, using some ideas of [2]; the proof will appear.

REMARK 5. As stated in [3], it was M. O. Rabin who first proposed conjecture $(d, 0, k)$, for all $d \geqq 1$ and $k \geqq 1$.

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