ON THE CONJUGATING REPRESENTATION
OF A FINITE GROUP

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Abstract. It is shown that the sum of the elements of the
character table of a finite group is at least \# conjugacy classes +
(# involutions—# classes of involutions)+(# real classes—#
strongly real classes). Equality sometimes holds, e.g. for \(A_5\). Our
investigations also demonstrate the appearance of a nontrivial real
valued character (whose degree we can estimate) in the decomposi-
tion of the conjugating representation of a finite group possessing
noncentral involutions.

1. In this note we show that any group \(G\) of even order \(g\) which
possesses noncentral involutions also possesses a nontrivial real valued
irreducible character \(\gamma\) of type \(R\) which appears with positive multiplicity
in the decomposition of the conjugating representation \(v\) of \(G\) acting on
itself. We also obtain an upper bound on the degree of \(\gamma\).

In fact, if \(v = \sum a_x x\) is the decomposition of \(v\) into irreducible con-
stituents, we show that \(\sum \epsilon(x) a_x \geq c + (m—r) + (k_1—k_2)\) where \(c\) denotes
the number of conjugacy classes of \(G\), \(m\) the number of involutions, \(r\) the
number of classes of involutions, \(k_1\) the number of real conjugacy classes,
\(k_2\) the number of strongly real conjugacy classes (i.e. classes generated by
the identity or by a product of at most two involutions), and \(\epsilon(x) = 0, \pm 1\)
depending on whether \(x\) is of type \(C, R,\) or \(H\) (Frobenius-Schur [2]).
Our upper bound on \(\gamma(1)\) is \((g—c)/(m—r)+(k_1—k_2)\) (cf. Brauer-Fowler
[1, Theorem 5A]).

2. Consider the class function \(v^*\) defined on \(G\) as \(v^*(a) =\) the number
of \(Y \in G\) such that either \(a^{-1}Ya = Y\) or \(a^{-1}Ya = Y^{-1}\). We shall not make
use of this fact, but \(v^*\) is the character of a permutation representation of
\(G/Z\) (in fact, if \(G\) acts faithfully on a finite set \(S\), and \(\sim\) is an equivalence
relation on \(S\) which is preserved by \(G\), then \(\delta^*(a) = \) the number of \(s \in S\)
such that \(sa \sim s\) is the character of a permutation representation of \(G\).

Observe

\[ v^*(a) \leq |C(a^2)|, \]
the order of the centralizer of \( a^2 \);

\[ v^*(a) = |C(a)| + t(a^2) - (v_a + 1) \]

where \( t(a^2) \) is the number of solutions in \( G \) of \( y^2 = a^2 \), and \( v_a \) is the number of involutions in \( C(a) \) (observe \( y = ax \) satisfies \( y^2 = a^2 \) if and only if \( x^2 = x^{-1} \)); in particular, if \( a \) is an involution, then \( v^*(a) = |C(a)| + m - v_a \). Hence we have

\[ \frac{1}{g} \sum_{a \in G} (|C(a^2)| - v^*(a)) \geq 0, \]

and in fact

\[ \frac{1}{g} \sum_{a \in G} (|C(a^2)| - v^*(a)) \geq \frac{1}{g} \sum_{a \in C(a)} (g - |C(a)| - m + v_a). \]

We now compute both sides of (4). In what follows, \( a_i \) denotes an element of the \( i \)th conjugacy class \( (a_0 = e, a_1, \ldots, a_r \) are involutions, \( a_{r+1}, \ldots, a_{k-1} \) are strongly real), \( n_i = |C(a_i)| \), and \( c_i \) is the number of involutions which invert \( a_i \), \( i \geq r + 1 \). We have

\[ \sum_{i=1}^{r} e(x) a_i - (c + k_1 - r - 1) \geq m - r - \frac{c_i - k_i}{g} + \sum_{i=1}^{r} \frac{v_i}{n_i} \cdot \]

By Brauer-Fowler [1], equation (4):

\[ \frac{m^2}{g} = \frac{m}{g} + \sum_{i=1}^{r} \frac{v_i - 1}{n_i} + \sum_{i=r+1}^{k-1} \frac{c_i}{n_i} \]

and so the right-hand side of (5) is

\[ = m - r - \frac{c_i - k_i}{g} + \sum_{i=1}^{r} \frac{v_i}{n_i} \geq m - r - (k_2 - r - 1). \]

Hence \( \sum_{x \neq 1} e(x) a_x \geq c + (m-r) + (k_1 - k_2) \), or

\[ \sum_{x \neq 1} e(x) a_x \geq (m - r) + (k_1 - k_2). \]

Since \( m > r \) whenever there are involutions outside of the center of \( G \), we must have under these circumstances at least one character \( x \) such that \( e(x) = 1 \) and \( a_x > 0 \). If we let \( \gamma \) denote the character of this type of smallest degree, then

\[ \sum_{x \neq 1} e(x) a_x \leq \sum_{e(x) = 1; x \neq 1} a_x \leq \sum_{e(x) = 1; x \neq 1} a_x \frac{x(1)}{\gamma(1)} \leq \frac{g - c}{\gamma(1)} \]

whence \((g - c)/((m - r) + (k_1 - k_2)) \geq \gamma(1)\).
3. We conclude with three remarks:

(1) Solomon [3] has observed that $a_x$ is the sum of the elements of the $x$th row of the character table of $G$. Since $m = \sum_{x \neq 1} \varepsilon(x)x(1)$ (Frobenius-Schur [2]), we have the following relationship concerning the elements of the character table outside the first row and first column:

\[ \sum_{x \neq 1; j \neq 1} \varepsilon(x)x(a_j) \geq k_1 - k_2 - r. \]

Equality holds for $A_5$, for example, so we cannot in general replace $\geq$ by $>$. 

(2) If $m - k_2 + 1 < 0$ it is more advantageous to simply compute inequality (3) above. It seems probable though, at least for groups with no normal 2-subgroups, that $m \geq k_2$.

(3) An upper bound for $\sum a_x$ is

\[ \sum a_x^2 - c^2 + c = \frac{1}{g} \sum |C(a)|^2 - c^2 + c. \]

Hence, writing our inequalities together,

\[ \sum n_i - c^2 + c \geq \sum a_x \geq \sum \varepsilon(x)a_x \geq c + (m - r) + (k_1 - k_2). \]

REFERENCES

