PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, July 1972

ON THE CONJUGATING REPRESENTATION OF A FINITE GROUP

K. L. FIELDS[1]

ABSTRACT. It is shown that the sum of the elements of the character table of a finite group is at least # conjugacy classes+(# involutions−# classes of involutions)+(# real classes−# strongly real classes). Equality sometimes holds, e.g. for A_5. Our investigations also demonstrate the appearance of a nontrivial real valued character (whose degree we can estimate) in the decomposition of the conjugating representation of a finite group possessing noncentral involutions.

1. In this note we show that any group G of even order g which possesses noncentral involutions also possesses a nontrivial real valued irreducible character γ of type $\boldsymbol{R}$ which appears with positive multiplicity in the decomposition of the conjugating representation ν of G acting on itself. We also obtain an upper bound on the degree of γ.

In fact, if $\nu=\sum a_x x$ is the decomposition of ν into irreducible constituents, we show that $\sum \varepsilon(x)a_x \geqq c+(m-r)+(k_1-k_2)$ where c denotes the number of conjugacy classes of G, m the number of involutions, r the number of classes of involutions, k_1 the number of real conjugacy classes, k_2 the number of strongly real conjugacy classes (i.e. classes generated by the identity or by a product of at most two involutions), and $\varepsilon(x)=0, \pm 1$ depending on whether x is of type $\boldsymbol{C}$, $\boldsymbol{R}$, or $\boldsymbol{H}$ (Frobenius-Schur [**2**]). Our upper bound on $\gamma(1)$ is $(g-c)/((m-r)+(k_1-k_2))$ (cf. Brauer-Fowler [**1**, Theorem 5A]).

2. Consider the class function ν^* defined on G as $\nu^*(a)$=the number of $Y \in G$ such that either $a^{-1}Ya=Y$ or $a^{-1}Ya=Y^{-1}$. We shall not make use of this fact, but ν^* is the character of a permutation representation of G/Z (in fact, if G acts faithfully on a finite set S, and $\sim$ is an equivalence relation on S which is preserved by G, then $\delta^*(a)$=the number of $s \in S$ such that $s^a \sim s$ is the character of a permutation representation of G). Observe

$$\nu^*(a) \leqq |C(a^2)|, \tag{1}$$

Received by the editors October 26, 1971.

AMS 1969 *subject classifications*. Primary 2080.

[1] Research supported by National Science Foundation grant GP-28700.

the order of the centralizer of a^2;

$$\nu^*(a) = |C(a)| + t(a^2) - (\nu_a + 1) \tag{2}$$

where $t(a^2)$ is the number of solutions in G of $Y^2=a^2$, and ν_a is the number of involutions in $C(a)$ (observe $y=ax$ satisfies $y^2=a^2$ if and only if $x^a=x^{-1}$); in particular, if a is an involution, then $\nu^*(a)=|C(a)|+m-\nu_a$. Hence we have

$$\frac{1}{g}\sum (|C(a^2)| - \nu^*(a)) \geqq 0, \tag{3}$$

and in fact

$$\frac{1}{g}\sum_{a\in G}(|C(a^2)| - \nu^*(a)) \geqq \frac{1}{g}\sum_{a^2=1}(g - |C(a)| - m + \nu_a). \tag{4}$$

We now compute both sides of (4). In what follows, a_i denotes an element of the ith conjugacy class ($a_0=e, a_1, \cdots, a_r$ are involutions, $a_{r+1}, \cdots, a_{k_2-1}$ are strongly real), $n_i=|C(a_i)|$, and c_i is the number of involutions which invert a_i, $i\geqq r+1$. We have

$$\sum \varepsilon(x)a_x - (c + k_1 - r - 1) \geqq m - r - \frac{m^2}{g} + \sum_{i=1}^{r}\frac{\nu_i}{n_i}. \tag{5}$$

By Brauer-Fowler [1], equation (4):

$$\frac{m^2}{g} = \frac{m}{g} + \sum_{i=1}^{r}\frac{\nu_i - 1}{n_i} + \sum_{i=r+1}^{k_2-1}\frac{c_i}{n_i}$$

and so the right-hand side of (5) is

$$= m - r - \frac{m}{g} - \sum_{i=1}^{r}\frac{\nu_i - 1}{n_i} + \sum_{i=1}^{r}\frac{\nu_i}{n_i} - \sum_{i=r+1}^{k_2-1}\frac{c_i}{n_i}$$

$$= m - r - \sum_{i=r+1}^{k_2-1}\frac{c_i}{n_i} \geqq m - r - (k_2 - r - 1).$$

Hence $\sum \varepsilon(x)a_x \geqq c+(m-r)+(k_1-k_2)$, or

$$\sum_{x\neq 1}\varepsilon(x)a_x \geqq (m - r) + (k_1 - k_2). \tag{6}$$

Since $m>r$ whenever there are involutions outside of the center of G, we must have under these circumstances at least one character x such that $\varepsilon(x)=1$ and $a_x>0$. If we let γ denote the character of this type of smallest degree, then

$$\sum_{x\neq 1}\varepsilon(x)a_x \leqq \sum_{\varepsilon(x)=1;x\neq 1} a_x \leqq \sum_{\varepsilon(x)=1;x\neq 1} a_x\frac{x(1)}{\gamma(1)} \leqq \frac{g - c}{\gamma(1)}$$

whence $(g-c)/((m-r)+(k_1-k_2))\geqq\gamma(1)$.

3. We conclude with three remarks:

(1) Solomon [3] has observed that a_x is the sum of the elements of the xth row of the character table of G. Since $m = \sum_{x\neq 1} \varepsilon(x)x(1)$ (Frobenius-Schur [2]), we have the following relationship concerning the elements of the character table outside the first row and first column:

$$\sum_{x\neq 1;\, j\neq 1} \varepsilon(x)\, x(a_j) \geqq k_1 - k_2 - r. \tag{7}$$

Equality holds for A_5, for example, so we cannot in general replace $\geqq$ by $>$.

(2) If $m-k_2+1<0$ it is more advantageous to simply compute inequality (3) above. It seems probable though, at least for groups with no normal 2-subgroups, that $m \geqq k_2$.

(3) An upper bound for $\sum a_x$ is

$$\sum a_x^2 - c^2 + c = \frac{1}{g}\sum |C(a)|^2 - c^2 + c.$$

Hence, writing our inequalities together,

$$\sum n_i - c^2 + c \geqq \sum a_x \geqq \sum \varepsilon(x)a_x \geqq c + (m-r) + (k_1 - k_2).$$

References

1. R. Brauer and K. A. Fowler, *On groups of even order*, Ann. of Math. (2) **62** (1955), 565–583. MR **17**, 580.

2. G. Frobenius and I. Schur, *Über die reelen Darstellungen der endlichen Gruppen*, S.-B. Preuss. Akad. Wiss. **1906**, 186–208.

3. L. Solomon, *On the sum of the elements in the character table of a finite group*, Proc. Amer. Math. Soc. **12** (1961), 962–963. MR **24** #A2619.

Department of Mathematics, University of Chicago, Chicago, Illinois 60637