

ON THE CONJUGATING REPRESENTATION OF A FINITE GROUP

K. L. FIELDS¹

ABSTRACT. It is shown that the sum of the elements of the character table of a finite group is at least $\#$ conjugacy classes + $(\#$ involutions $- \#$ classes of involutions) + $(\#$ real classes $- \#$ strongly real classes). Equality sometimes holds, e.g. for A_5 . Our investigations also demonstrate the appearance of a nontrivial real valued character (whose degree we can estimate) in the decomposition of the conjugating representation of a finite group possessing noncentral involutions.

1. In this note we show that any group G of even order g which possesses noncentral involutions also possesses a nontrivial real valued irreducible character γ of type R which appears with positive multiplicity in the decomposition of the conjugating representation ν of G acting on itself. We also obtain an upper bound on the degree of γ .

In fact, if $\nu = \sum a_x x$ is the decomposition of ν into irreducible constituents, we show that $\sum \varepsilon(x) a_x \geq c + (m - r) + (k_1 - k_2)$ where c denotes the number of conjugacy classes of G , m the number of involutions, r the number of classes of involutions, k_1 the number of real conjugacy classes, k_2 the number of strongly real conjugacy classes (i.e. classes generated by the identity or by a product of at most two involutions), and $\varepsilon(x) = 0, \pm 1$ depending on whether x is of type C , R , or H (Frobenius-Schur [2]). Our upper bound on $\gamma(1)$ is $(g - c) / ((m - r) + (k_1 - k_2))$ (cf. Brauer-Fowler [1, Theorem 5A]).

2. Consider the class function ν^* defined on G as $\nu^*(a) =$ the number of $Y \in G$ such that either $a^{-1}Ya = Y$ or $a^{-1}Ya = Y^{-1}$. We shall not make use of this fact, but ν^* is the character of a permutation representation of G/Z (in fact, if G acts faithfully on a finite set S , and \sim is an equivalence relation on S which is preserved by G , then $\delta^*(a) =$ the number of $s \in S$ such that $s^a \sim s$ is the character of a permutation representation of G). Observe

$$(1) \quad \nu^*(a) \leq |C(a^2)|,$$

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the order of the centralizer of a^2 ;

$$(2) \quad \nu^*(a) = |C(a)| + t(a^2) - (\nu_a + 1)$$

where $t(a^2)$ is the number of solutions in G of $Y^2=a^2$, and ν_a is the number of involutions in $C(a)$ (observe $y=ax$ satisfies $y^2=a^2$ if and only if $x^a=x^{-1}$); in particular, if a is an involution, then $\nu^*(a)=|C(a)|+m-\nu_a$. Hence we have

$$(3) \quad \frac{1}{g} \sum (|C(a^2)| - \nu^*(a)) \geq 0,$$

and in fact

$$(4) \quad \frac{1}{g} \sum_{a \in G} (|C(a^2)| - \nu^*(a)) \geq \frac{1}{g} \sum_{a^2=1} (g - |C(a)| - m + \nu_a).$$

We now compute both sides of (4). In what follows, a_i denotes an element of the i th conjugacy class ($a_0=e, a_1, \dots, a_r$ are involutions, $a_{r+1}, \dots, a_{k_2-1}$ are strongly real), $n_i=|C(a_i)|$, and c_i is the number of involutions which invert $a_i, i \geq r+1$. We have

$$(5) \quad \sum \varepsilon(x)a_x - (c + k_1 - r - 1) \geq m - r - \frac{m^2}{g} + \sum_{i=1}^r \frac{\nu_i}{n_i}.$$

By Brauer-Fowler [1], equation (4):

$$\frac{m^2}{g} = \frac{m}{g} + \sum_{i=1}^r \frac{\nu_i - 1}{n_i} + \sum_{i=r+1}^{k_2-1} \frac{c_i}{n_i}$$

and so the right-hand side of (5) is

$$\begin{aligned} &= m - r - \frac{m}{g} - \sum_{i=1}^r \frac{\nu_i - 1}{n_i} + \sum_{i=1}^r \frac{\nu_i}{n_i} - \sum_{i=r+1}^{k_2-1} \frac{c_i}{n_i} \\ &= m - r - \sum_{i=r+1}^{k_2-1} \frac{c_i}{n_i} \geq m - r - (k_2 - r - 1). \end{aligned}$$

Hence $\sum \varepsilon(x)a_x \geq c + (m-r) + (k_1-k_2)$, or

$$(6) \quad \sum_{x \neq 1} \varepsilon(x)a_x \geq (m - r) + (k_1 - k_2).$$

Since $m > r$ whenever there are involutions outside of the center of G , we must have under these circumstances at least one character x such that $\varepsilon(x)=1$ and $a_x > 0$. If we let γ denote the character of this type of smallest degree, then

$$\sum_{x \neq 1} \varepsilon(x)a_x \leq \sum_{\varepsilon(x)=1, x \neq 1} a_x \leq \sum_{\varepsilon(x)=1, x \neq 1} a_x \frac{x(1)}{\gamma(1)} \leq \frac{g - c}{\gamma(1)}$$

whence $(g-c)/((m-r)+(k_1-k_2)) \geq \gamma(1)$.

3. We conclude with three remarks:

(1) Solomon [3] has observed that a_x is the sum of the elements of the x th row of the character table of G . Since $m = \sum_{x \neq 1} \varepsilon(x)x(1)$ (Frobenius-Schur [2]), we have the following relationship concerning the elements of the character table outside the first row and first column:

$$(7) \quad \sum_{x \neq 1; j \neq 1} \varepsilon(x) x(a_j) \geq k_1 - k_2 - r.$$

Equality holds for A_5 , for example, so we cannot in general replace \geq by $>$.

(2) If $m - k_2 + 1 < 0$ it is more advantageous to simply compute inequality (3) above. It seems probable though, at least for groups with no normal 2-subgroups, that $m \geq k_2$.

(3) An upper bound for $\sum a_x$ is

$$\sum a_x^2 - c^2 + c = \frac{1}{g} \sum |C(a)|^2 - c^2 + c.$$

Hence, writing our inequalities together,

$$\sum n_i - c^2 + c \geq \sum a_x \geq \sum \varepsilon(x)a_x \geq c + (m - r) + (k_1 - k_2).$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637