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ON WHITEHEAD PRODUCTS[1]

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ABSTRACT. For every $(n-1)$-connected space X, the image of the Whitehead product map $\pi_n(X)\times\pi_n(X)\to\pi_{2n-1}(X)$ is estimated in terms of the nth and the $2n$th Betti numbers.

Let X be a path connected space, and let p and q be positive integers. Denote by $[\pi_p, \pi_q](X)$ the subgroup of $\pi_{p+q-1}(X)$ generated by the image of the Whitehead product map

$$\pi_p(X) \times \pi_q(X) \to \pi_{p+q-1}(X).$$

The purpose of this note is to prove briefly the following result:

THEOREM 2. *Let X be $(n-1)$-connected, $n\geqq 1$, and let $b_r=\dim_k H^r(X; k)$, k being a field of characteristic $\neq 2$. Set $b=\frac{1}{2}b_n(b_n-1)$ or $\frac{1}{2}b_n(b_n+1)$ according as n is odd or even. If $b>b_{2n}$, then $[\pi_n, \pi_n](X)$ is nontrivial. If, moreover, $n>1$, then the vector space $[\pi_n, \pi_n](X)\otimes_Z k$ is of dimension $\geqq b-b_{2n}$.*

Let ϕ denote the Hurewicz homomorphism. Let k be a commutative ring with 1. Throughout this note, k will be used for the coefficient of cohomology. There is a pairing

$$H^*(X) \otimes_k H^*(X) \times H_*(X) \otimes_Z H_*(X) \to k \tag{1}$$

such that, for $\bar{w}' \in H^p(X)$, $\bar{w}'' \in H^q(X)$, $z' \in H_p(X)$, $z'' \in H_q(X)$, $\langle \bar{w}'\otimes_k\bar{w}'', z'\otimes_Z z''\rangle=\langle\bar{w}', z'\rangle\langle\bar{w}'', z''\rangle$, when $p\neq q$, and $=\langle\bar{w}', z'\rangle\langle\bar{w}'', z''\rangle+(-1)^{pq}\langle\bar{w}', z''\rangle\langle\bar{w}'', z'\rangle$, when $p=q$.

Denote by $\mathrm{Hom}([\pi_p, \pi_q](X), k)$ the k-module of homomorphisms from the group $[\pi_p, \pi_q](X)$ to the additive group of k. Theorem 2 is a consequence of the next assertion.

THEOREM 1. *If N is the kernel of the cup product map*

$$H^p(X) \otimes_k H^q(X) \to H^{p+q}(X),$$

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then there is a k-module homomorphism

$$g:N \to \mathrm{Hom}([\pi_p, \pi_q](X); k)$$

sending ξ to g_ξ such that

$$g_\xi([[\alpha'], [\alpha'']]) = \langle \xi, \phi[\alpha'] \otimes_Z \phi[\alpha''] \rangle.$$

PROOF. We are going to give, in essence, a singular cohomological version of the proof for a more restricted result for fundamental groups given in [**1**]. Our basic tools are the functional product of N. Steenrod [**4**], and a result of H. Uehara and W. S. Massey [**3**].

Choose a base point of X. For $\xi = \sum \bar{w}_i' \otimes_k \bar{w}_i'' \in N$, choose $w_i' \in Z^p(X, x_0)$ and $w_i'' \in Z^q(X, x_0)$ representing respectively $\bar{w}_i'$ and $\bar{w}_i''$ such that

$$\sum w_i' \cup w_i'' + \delta w = 0$$

for some $w \in C^{p+q-1}(X, x_0)$. Given a map $\alpha:(I^m, \mathring{I}^m) \to (X, x_0)$, $m \geqq 1$, choose $v_i \in C^{p-1}(I^m, 0)$ with $\delta v_i = \alpha^* w_i'$. Observe that $v_i' \cup \alpha^* w_i'' \in C^{p+q-1}(I^m, \mathring{I}^m)$. Modulo $\alpha^* H^{p+q-1}(X)$, the cohomology class $\bar{u}$ of $u = \sum v_i \cup \alpha^* w_i'' + \alpha^* w \in Z^{p+q-1}(I^m, \mathring{I}^m)$ is uniquely determined by ξ. Thus there is a k-module homomorphism

$$\lambda_\alpha : N \to H^{p+q-1}(I^m, \mathring{I}^m)/\alpha^* H^{p+q-1}(X).$$

Verify that λ_α depends only on the homotopy class $[\alpha]$ and is trivial when $m \neq p+q-1$. The homomorphism λ_α can be considered as the Steenrod functional product with respect to the map α arising from the relation $\sum \bar{w}_i' \cup \bar{w}_i'' = 0$.

Let $\mathring{I}^m$ be oriented, and let z_m be the generator of $H_m(I^m, \mathring{I}^m)$. Define

$$g:N \to \mathrm{Hom}([\pi_p, \pi_q](X); k)$$

such that, for $[\alpha] \in [\pi_p, \pi_q](X)$,

$$g_\xi([\alpha]) = \langle \bar{u}, z_m \rangle, \qquad m = p + q - 1.$$

Since $\phi[\pi_p, \pi_q](X) = 0$, g_ξ is well defined and can be shown to be a homomorphism. It follows from a slight modification of Theorem IV [**3**] that

$$g_\xi([[\alpha'], [\alpha'']]) = \langle \xi, \phi[\alpha'] \otimes_Z \phi[\alpha''] \rangle.$$

Hence the theorem is proved.

PROOF OF THEOREM 2. We have $p = q = n$ and a field k of characteristic $\neq 2$. Let the pairing

$$H^n(X) \otimes_k H^n(X) \times H_n(X) \otimes_Z H_n(X) \to k$$

be the restriction of the pairing (1). We have an induced linear map

$$h: H^n(X) \otimes_k H^n(X) \to \operatorname*{Hom}_Z(H_n(X) \otimes_Z H_n(X), k)$$

whose image is denoted by B. Then $\dim_k B = b$.

Write $N' = hN \subset B$. The cup product of $H^n(X)$ has a factorization

$$H^n(X) \otimes_k H^n(X) \xrightarrow{h} B \to H^{2n}(X),$$

and N' is the kernel of the map $B \to H^{2n}(X)$. It follows that $\dim_k N' \geqq b - b_{2n}$. There is an isomorphism $gN \approx N'$ with $g_\xi \mapsto g'_\xi$ such that

$$g'_\xi(\phi[\alpha'] \otimes_Z \phi[\alpha'']) = g_\xi([[\alpha'], [\alpha'']]).$$

Consequently,

$$\dim_k \operatorname{Hom}([\pi_n, \pi_n](X), k) \geqq \dim_k N' \geqq b - b_{2n}.$$

For $n > 1$, $\dim_k [\pi_n, \pi_n](X) \otimes_Z k$ and $\dim_k \operatorname{Hom}([\pi_n, \pi_n](X), k)$ are either equal or both ∞.

COROLLARY. *If X is $(n-1)$-connected, $n > 1$, and if $b_{2n} = 0$, then*

$$\dim_k [\pi_n, \pi_n](X) \otimes_Z k = b,$$

provided k is of characteristic $\neq 2$.

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