DEFICIENCY IN F-MANIFOLDS

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Abstract. Let $M$ be a manifold modelled on a metrizable, locally-convex, topological vector space $F$ such that $F \cong F^m$, and let $K$ be a closed subset of $M$. Then the following are equivalent: (1) $K$ is locally a subset of a collared submanifold of $M$, (2) each $x \in K$ has an open neighborhood $U$ and a homeomorphism $h: U \rightarrow I_2 \times F$ such that $h(U \cap K) \subset \{0\} \times F$, (3) each $x \in K$ has an open neighborhood $U$ and a homeomorphism $h: U \rightarrow F \times F$ such that $h(U \cap K) \subset \{0\} \times F$, (4) there is a homeomorphism $h: M \rightarrow M \times F$ such that for $x \in K$, $h(x) = (x, 0)$, (5) $K$ is infinite-deficient (i.e. there is a homeomorphism $h: M \rightarrow M \times F$ such that $h(K) \subset M \times \{0\}$), and (6) $K$ is the finite union of sets each having one of the above properties.

1. Introduction. Let $F$ be a metrizable, locally-convex, topological vector space (MLCTVS) such that $F \cong F^m$ ($F$ is homeomorphic to the countable product of itself). An $F$-manifold is a paracompact Hausdorff space with a covering of open sets homeomorphic to $F$. Deficient subsets of $F$-manifolds are essentially those which are contained in subspaces of infinite codimension. Anderson [1] was the first to do work in this area, and he was concerned at that time with separable Fréchet spaces (which are all homeomorphic to $s$, the countable product of lines, and $l_2$, separable Hilbert space, see [2]).

Recent results have tied up the notions of deficiency with that of negligibility ($Y \subset X$ is negligible if $X \cong X - Y$) and that of Property Z (homotopic negligibility). (See [3] and [4].) This paper characterizes the concept of deficiency for closed subsets of $F$-manifolds.

Let $M$ be an $F$-manifold and let $K$ be a closed subset of $M$. $K$ is locally infinite-deficient (l.i.d.) if each point in $K$ has an open neighborhood $U$ and a homeomorphism $h: U \rightarrow F \times l_2$ such that $h(U \cap K) \subset F \times \{0\}$. $K$ is locally $F$-deficient if each point of $K$ has an open neighborhood $U$ and a

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1 This paper is part of the author’s dissertation done under David W. Henderson.

2 The condition “locally-convex” can be replaced by “$F \cong R \times Z$ for some metric space $Z$”. The author knows of no MTVS’s where this is not true.
homeomorphism $h: U \to F \times F$ such that $h(U \cap K) \subseteq F \times \{0\}$, $K$ is $F$-deficient if there is a homeomorphism $h: M \to M \times F$ such that $h(K) \subseteq M \times \{0\}$. Finally, $K$ is infinite-deficient (as it is usually defined in the literature) if $K$ is $I_2$-deficient. It should be noted that by Schori [8] homeomorphisms of $M$ onto $M \times F$ do exist. The main result of this paper is that all four definitions are equivalent.

2. The main theorems.

**Theorem 2.1.** Let $K$ be a closed subset of an $F$-manifold $M$, where $F$ is a $MLCTVS$ such that $F \cong F^\omega$. Then the following are equivalent:

1. $K$ is locally a subset of a collared submanifold of $M$,
2. $K$ is locally infinite-deficient,
3. $K$ is locally $F$-deficient,
4. $K$ is $F$-deficient,
5. $K$ is infinite-deficient,
6. $K$ is the finite union of sets each of which satisfies one of the above.

**Proof.** Klee was the first to show that $l_2 \times [0, 1)$ is homeomorphic to $l_2$ (Theorem III (1.3) of [6]). Since $F$ is locally convex, it is known that $F \cong F^\omega \times Z$ where $Z$ is some metric space. Using $F^\omega \cong F^\omega \times I_2$ it follows easily that $F \cong F \times l_2$. This combined with the first statement shows that $(2) \Rightarrow (1)$. Again using $F \cong F \times l_2$, it follows that $(4) \Rightarrow (5)$ and $(3) \Rightarrow (2)$, for if $(K, M) \subseteq ([0] \times [0] \times M, l_2 \times F \times M) \subseteq ([0] \times F \times M, l_2 \times F \times M) = ([0] \times M, l_2 \times M)$. Also $(5) \Rightarrow (2)$ and $(4) \Rightarrow (3)$ are obvious. That $(1) \Rightarrow (3)$ follows from Proposition 5.1 of §5. Finally, Proposition 5.3 of §5 shows that $(6) \Rightarrow (4)$ when the sets in the finite set satisfy property (3).

**Theorem 2.2.** Let $F$ and $M$ satisfy the hypotheses of Theorem 2.1, and let $K$ satisfy any of the equivalent conditions in the theorem. Then there is a homeomorphism $h: M \to M \times F$ such that for $x \in K$, $h(x) = (x, 0)$.

**Proof.** Apply Theorem 2.1 and Proposition 5.3.

3. Definitions. By the cone over $X$, we shall mean $X \times I$ with the points $X \times \{0\}$ identified, with the following topology: At the vertex, a basis for the open sets will be the sets $\{(x, t) \in X \times I | t < s\}$ for each $s \in (0, 1)$, and at other points the product topology will be used. The open cone, denoted $C(X)$, will be the cone minus $X \times \{1\}$.

By Lemma 2 of [5], $F \cong C(E)$ for some metric space $E$. Let $h: F \to C(E)$ be such a homeomorphism. For $y \in F$, define $\|y\| = \pi_2 h(y)$, that is, the second coordinate of $h(y)$. If $\|y\| = 0$, we call $y$ the 0-element in $F$. Let $B_t = \{y \in F | \|y\| < t\}$.
Let \( X \) be a topological space, \( F \) a MLCTS, and \( r: X \to I \) a continuous function. Then the variable product of \( X \) by \( F \) with respect to \( r \) is defined to be

\[
X \times_r F = \{(x, y) \in X \times F \mid y = 0 \text{ or } \|y\| < r(x)\}.
\]

We say that this variable product is zero over \( r^{-1}(0) \). For \( U \) a subset of \( X \), the fiber over \( U \) is the set \((U \times F) \cap (X \times_r F)\), and may also be called the variable product restricted to \( U \).

4. **Isotopy lemmas.** The following technical lemmas construct isotopies which prove to be quite useful in pushing around deficient sets. All of the isotopies will be invertible (have continuous inverses).

**Lemma 4.1 (Schori).** There exists an isotopy \( h_t: F \times F \to F \times F, \; t \in I \), satisfying

1. \( h_0 = \text{id} \),
2. for \( t < 1 \), \( h_t \) is onto \( B_{(1-t)} \times F \),
3. \( h_t \) is onto \( \{0\} \times F \).

The proof is due to Anderson and Schori (see Lemma 5.3 of [8]).

**Lemma 4.2.** There exists a double isotopy \( r_{f_t}: F \to F, \; r \in (0, 1], \; t \in [0, 1+\frac{r}{2}] \) satisfying

1. \( r_{f_0} = \text{id} \),
2. for \( t < 1 \), \( r_{f_t} \) is a homeomorphism,
3. \( r_{f_1} \) is onto \( F - \{0\} \),
4. for \( t > 1 \), \( r_{f_t} \) is onto \( f - B_{(1-1)} \),
5. \( r_{f_t} \) is the identity outside \( B_r \).

**Proof.** For \( t \leq 1 \), the isotopy is constructed using the isotopy pushing the origin off \( l_2 \) which is defined by Anderson and Bing [2, pp. 784–786] and the fact that \( F \) has \( l_2 \) as a topological factor. For \( t \geq 1 \), the isotopy is easily constructed using \( \|y\| \) for \( y \in F \).

**Lemma 4.3.** There exists a double isotopy \( r_{g_t}: F \times F \to F \times F, \; r \in (0, 1], \; t \in I, \) satisfying

1. \( r_{g_0} = \text{id} \),
2. for \( t < 1 \), \( r_{g_t} \) is onto a variable product of \( F \) by \( F \) which is nowhere zero,
3. \( r_{g_t} \) is onto a variable product of \( F \) by \( F \) which is zero over \( \{0\} \),
4. \( r_{g_t} = \text{id} \) outside \( B_r \times F \) and on \( F \times \{0\} \).

**Proof.** Using the isotopy of Lemma 4.2, define

\[
r_{g_t}(x, y) = (r_{f_t}(x), y)
\]
where \( r_1 \) and \( t_1 \) are functions defined by

\[
\begin{align*}
  r_1 &= \min \{ r, \| y \| \}, \\
  t_1 &= 0 \quad \text{for } y = 0, \\
  &= \frac{1}{2} \left[ (1 + r/2) - r/2(\| y \| + 1) \right] \quad \text{for } \| y \| \geq (1 - t)/t, \\
  &= \max \{ 0, \| y \| + 1 - (1 - t)/t \} \quad \text{for } \| y \| \leq (1 - t)/t, \quad (1 - t)/t \geq 1, \\
  &= \frac{t \| y \|}{(1 - t)} \quad \text{for } \| y \| \leq (1 - t)/t \leq 1.
\end{align*}
\]

5. Three propositions.

**Proposition 5.1.** \( ([0, 1] \times F, \{ 0 \} \times F) \) and \( (F \times F, \{ 0 \} \times F) \) are homeomorphic as pairs.

**Proof.** Regard the \( F \) in the first pair as being \( F \times F \). Using the isotopy of Lemma 4.1, define \( f: [0, 1] \times F \times F \to [0, 1] \times F \times F \) by \( f(s, x, y) = (s, h(s, y)(x, y)) \).

It is clear that \( f \) is an embedding onto a variable product of \( [0, 1] \times F \) by \( F \) which is zero over \( \{ 0 \} \times F \), and this variable product is easily seen to be homeomorphic to the open cone of \( F \) crossed with \( F \). Furthermore, \( f \) takes the set \( \{ 0 \} \times F \times F \) onto the vertex of the cone crossed with \( F \). Since \( F \) is homeomorphic to its unit sphere (use \( F \times ([0, 1] \times F) \) to show that the unit sphere is homeomorphic to \( F - \{ 0 \} \) and fill in the origin using Lemma 4.2), it follows that the open cone over \( F \) is homeomorphic to \( F \).

**Proposition 5.2.** Let \( M \) be an \( F \)-manifold, and let \( K \) be a closed locally \( F \)-deficient subset of \( M \). Let \( p : M \to (0, 1] \) be a continuous function. Then there is a continuous function \( r : M \to I \) and a homeomorphism \( h : M \times F \to M \times F \) such that \( r^{-1}(0) = K \), \( h = \text{id} \) on \( M \times \{ 0 \} \), and for \( (x, y) \in M \times F \),

\[
d(x, r(x)h((x, y))) < p(x),
\]

where \( d \) is the metric on \( M \).

The proposition does not follow from [8], as might be suspected. The condition "\( h = \text{id} \) on \( M \times \{ 0 \} \)" is crucial. The proof makes use of "lateral" pushes rather than "vertical" ones as in [8].

**Proof.** We use a theorem due to E. Michael [7] which is quite useful.

**Theorem (Michael).** Let \( P \) be a topological property satisfying

(a) if \( U \) has Property \( P \), then every open subset of \( U \) has Property \( P \),

(b) if \( U \) is the union of two open (in \( U \)) sets both of which have Property \( P \), then \( U \) has Property \( P \),

(c) if \( U \) is the union of disjoint open (in \( U \)) sets, each of which has Property \( P \), then \( U \) has Property \( P \).

If \( X \) is a paracompact Hausdorff space having Property \( P \) locally, then \( X \) has Property \( P \).

**Definition.** An open subset \( U \) of \( M \) has Property \( P \) if given \( W \) an open set in \( M \), \( C \) and \( D \) closed subsets of \( K \), and \( p : M \to (0, 1] \) and \( r_0 : M \to I \)
continuous functions such that $C \subseteq W \subseteq U$ and $r_0^{-1}(0) = D$; then there is a continuous function $r_1: M \to I$ and a homeomorphism $h: M \times r_0 F \to M \times r_1 F$ such that

1. $r_1 \leq r_0$ and $r_1^{-1}(0) = D \cup C$,
2. $h[M \times \{0\}] = \text{id}$,
3. $h[M - W] \times r_0 F = \text{id}$,
4. for $(x, y) \in M \times r_0 F$, $d(x, \pi_1 h((x, y))) < p(x)$.

That Property P satisfies conditions (a) and (c) of Michael's theorem is easy to check. As is always the case, condition (b) is more complicated to show.

Let $U_1$ and $U_2$ be open subsets of $M$ having Property P and let $U = U_1 \cup U_2$. Let $C, W, D, p$ and $r_0$ be as in the definition of Property P. For $i = 1, 2$, pick open sets $U'_i$ and $U''_i$ such that $U'_i \subseteq U_i$ and $U''_i \subseteq U_i$ and $U'_i \cup U''_i = U$. For $i = 1, 2$, let $W_i = U'_i \cap W$ and $C_i = U''_i \cap C$. Let $D_1 = D$ and $D_2 = D \cup C_1$. Define $p^*: M \to (0, 1]$ by

$$p^*(x) = \frac{1}{2} \max \{\delta \in (0, 1] | p(z) \geq \delta \text{ for all } z \in N_\delta(x)\},$$

where $N_\delta(x)$ is the $\delta$-neighborhood of $x$.

Since $U_1$ has Property P, there exist a continuous function $r_1: M \to I$ and a homeomorphism $h_1: M \times r_0 F \to M \times r_1 F$ satisfying (1) $r_1 \leq r_0$ and $r_1^{-1}(0) = D_1 \cup C_1$, (2) $h_1[M \times \{0\}] = \text{id}$, (3) $h_1[(M - W_i) \times r_0 F] = \text{id}$, (4) for $(x, y) \in M \times r_0 F$, $d(x, \pi_1 h_1((x, y))) < p^*(x)$.

Similarly, since $U_2$ has Property P, there exist a continuous function $r_2: M \to I$ and a homeomorphism $h_2: M \times r_0 F \to M \times r_2 F$ satisfying (1) $r_2 \leq r_1$ and $r_2^{-1}(0) = D_2 \cup C_2$, (2) $h_2[M \times \{0\}] = \text{id}$, (3) $h_2[(M - W_2) \times r_1 F] = \text{id}$, (4) for $(x, y) \in M \times r_0 F$, $d(x, \pi_1 h_2((x, y))) < p^*(x)$.

Then the functions $r_2: M \to I$ and $h_2 \circ h_1: M \times r_0 F \to M \times r_2 F$ satisfy the requirements in the definition of Property P.

If we could show that $M$ has Property P, we would be finished, for we could let $C = W = U = M$ and $r_0(m) = 1$. Hence it suffices by Michael's theorem to show that $M$ has Property P locally. It is sufficient to show this when $U = F \times F$ and $K \cap U \subseteq F \times \{0\}$. Let $C, W, D, p$ and $r_0$ be as before. Let $F_1 = F \times \{0\}$. Let $r: F_1 \to I$ be defined by

$$r(x) = \min\{d((x, 0), W - D), p^*(x)\}.$$

Let $t: F_1 \to I$ be a continuous function such that $t^{-1}(1) = C - D$ and $t^{-1}(0) = (F_1 - D) - W$. Define $f: (F_1 - D) \times F \times F \to (F_1 - D) \times F \times F$ by

$$f(x, y, z) = (x, r(g_1(x), y, z)) \text{ if } r(x) \neq 0,$$

$$= \text{id} \text{ if } r(x) = 0,$$

where $g$ is the double isotopy of Lemma 4.3.
It is easily seen that $f$ is a homeomorphism onto a variable product of $(F_i - D) \times F$ by $F$ which is zero on $C - D$. Since the variable product of $(F_i - D) \times F$ by $F$ with respect to $r_0$ is nowhere zero, we can regard this product as $(F_i - D) \times F \times F$, and applying $f$ to this and extending by the identity to $(D \times F) \times r_0 F$, we will get a variable product and a homeomorphism satisfying the requirements in the definition of Property $P$.

**Proposition 5.3.** Let $M$ be an $F$-manifold, and let $K$ be a closed subset of $M$ which is the finite union of closed locally $F$-deficient sets. Then there is a homeomorphism $h: M \to M \times F$ such that for $x \in K$, $h(x) = (x, 0)$.

We may assume that $K = \bigcup_i K_i$ where each $K_i$ is closed and locally $F$-deficient. Using Proposition 5.2, there is a homeomorphism of $M \times F$ onto a variable product of $M$ by $F$ which is zero on $K_1$. Since this variable product restricted to $M - K_1$ is nonzero, we can regard the variable product over $M - K_1$ as just $(M - K_1) \times F$, and can apply Proposition 5.2 with the function $p$ as some continuous function approaching 0 at $K_1$, and we will get a homeomorphism onto a variable product which is zero on $K_2 - K_1$. Using the function $p$, one can show that this latter homeomorphism can be extended continuously by the identity to the set $K_1 \times \{0\}$. Then the composition of the two homeomorphisms gives a homeomorphism from $M \times F$ onto a variable product of $M$ by $F$ which is the identity on $M \times \{0\}$ and is zero on $K_1 \cup K_2$.

Repeated applications of Proposition 5.2 result in a homeomorphism of $M \times F$ onto a variable product of $M$ by $F$ which is zero over $K$ and is the identity on $M \times \{0\}$. Using the stability theorem of Schori (Corollary 2.3 of [8]) there is a homeomorphism $h: (M - K) \times F \to (M - K)$ satisfying

$$d(x, h((x, y))) < d(x, K)/2 \quad \text{for } (x, y) \in (M - K) \times F.$$  

Applying this homeomorphism to the variable product over $M - K$ (which, since it is nowhere zero, can be regarded as $(M - K) \times F$), one can extend continuously by the identity to $K \times \{0\}$, and the proof is completed.

**Bibliography**


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