ARITHMETIC STRUCTURE AND LACUNARY FOURIER SERIES

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Abstract. We prove a theorem concerning the arithmetic structure of \( \Lambda(p) \) sets. This generalizes a result of Rudin and yields a new characterization of Sidon sets for certain Abelian groups.

Let \( G \) be a compact Abelian group with dual group \( \Gamma \). A subset \( E \subseteq \Gamma \) is called a \( \Lambda(p) \) set (\( 2 < p < \infty \)) if for every trigonometric polynomial \( f = \sum_{n} a_{n} \gamma_{n} \) \( (\gamma_{1}, \ldots, \gamma_{n} \in E) \) an inequality \( \|f\|_{p} \leq B \|f\|_{2} \) holds, where the constant \( B \) depends only on \( E \) and \( p \). Theorems 1 and 2 of this paper give estimates on the number of elements of \( E \) which can lie in a given multidimensional arithmetic progression in \( \Gamma \). Theorem 2 is a generalization of Rudin's result [5, Theorem 3.5]. When a set \( E \) is \( \Lambda(p) \) for all \( 2 < p < \infty \) and the constant \( B \) may be taken as \( Cp^{1/2} \) where \( C \) depends only on \( E \) then it is called a \( \Lambda \) set. A subset \( E \subseteq \Gamma \) is said to be a Sidon set if every bounded function on \( E \) can be extended to be a Fourier-Stieltjes transform. It is known that every Sidon set is a \( \Lambda \) set, but the converse is unknown [4, p. 128]. As a corollary to Theorem 2 we find that \( \Lambda \) sets are also Sidon sets in groups \( \Gamma \) where each element has the same prime order.

Notation. Suppose \( s \) and \( N \) are positive integers, \( 1 \leq r < \infty \), \( b_{1}, \ldots, b_{k} \in \Gamma \) have finite order, \( b_{k+1}, \ldots, b_{s} \in \Gamma \) have infinite order and \( b_{0} \in \Gamma \) has arbitrary order. Set

\[
A_{r}(N, s, b) = \left\{ n \cdot b + b_{0} \left| \sum_{k+1}^{s} |n_{j}|^{r} \leq N^{r} \right\}
\]

where \( n \cdot b = n_{1}b_{1} + \cdots + n_{s}b_{s} \). Denote by \( \hat{f} \) the Fourier transform of \( f \in L^{1}(G) \), by \( \langle x, \gamma \rangle \) the action of the character \( \gamma \) on \( x \in G \), and by \( |F| \) the cardinality of a finite set \( F \).

**Theorem 1.** Let \( b_{1}, \ldots, b_{k} \in \Gamma \) have finite orders \( \beta_{1}, \ldots, \beta_{k} \), and let \( b_{k+1}, \ldots, b_{s} \in \Gamma \) have infinite order. Let \( E \subseteq \Gamma \) be a \( \Lambda(p) \) set with \( B \) a

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constant as in the definition of \( \Lambda(p) \) set. If \( A_2(N,s,b) = A_2 \) is as defined above and \( b_{k+1}, \ldots, b_s \) are independent in \( \Gamma \) then
\[
|A_2 \cap E| \leq B \epsilon^2 (\beta_1 \cdots \beta_s)^{2/p} (1 + N(\pi/2)^{1/2})^{(s-k)/p}.
\]

**Proof.** Since translates of \( \Lambda(p) \) sets are \( \Lambda(p) \) sets with the same constant of definition, we may take \( b_0 = 0 \). Define
\[
g_{b_j}(x) = \sum_{n=0}^{\beta_j-1} \langle x, nb_j \rangle \quad \text{for } j = 1, \ldots, k,
\]
\[
g_{b_j}(x) = \sum_{n \in \mathbb{Z}} \exp(-\|n^*/N\|^2) \langle x, nb_j \rangle \quad \text{for } j = k + 1, \ldots, s.
\]
It is easy to check that \( g_{b_j}(x) \geq 0 \) for all \( x \in G \).

Let \( \Gamma' = \mathbb{Z}(\beta_1) \oplus \cdots \oplus \mathbb{Z}(\beta_s) \oplus \mathbb{Z}^{s-k} \), then
\[
g(x) = \sum_{n \in \Gamma'} \exp(-\|n^*/N\|^2) \langle x, n \cdot b \rangle
\]
where \( n^* = (0, \ldots, 0, n_{k+1}, \ldots, n_s) \). The map \( \phi(n) = n \cdot b \) is a homomorphism of \( \Gamma' \) into \( \Gamma \). If \( H_y = \phi^{-1}(y) \) then
\[
g(y) = \sum_{n \in H_y} \exp(-\|n^*/N\|^2)
\]
when \( y \in \text{Im } \phi \) and \( g(y) = 0 \) otherwise.

We will show that
\[
(1) \quad g(0) \leq \epsilon g(y) \quad \text{for all } y \in A_2(N,s,b).
\]
Define an equivalence relation on \( H_0 \) by \( h_1 \) is related to \( h_2 \) if \( h_1 = \pm h_2 \). Denote the quotient space by \( \mathcal{H} \) and the class containing \( h \) by \( [h] \). Let \( y \in \text{Im } \phi \) be fixed and take \( n_y \in H_y \), then since \( H_y = n_y + H_0 \) we have
\[
g(y) = \sum_{h \in H_0} \exp(-\|m^* + n_y^*/N\|^2)
\]
\[
= \sum_{[m] \in \mathcal{H}} \sum_{m \in [m]} \exp(-\|m^*/N\|^2 - \|n_y^*/N\|^2 - 2m^* \cdot n_y^*/N^2)
\]
\[
= \exp(-\|n_y^*/N\|^2) \left( 1 + 2 \sum_{[m] \in \mathcal{H} - \{0\}} \exp(-\|m^*/N\|^2) \cosh(2m^* \cdot n_y^*/N^2) \right).
\]
Hence
\[
g(0) = 1 + 2 \sum_{[m] \in \mathcal{H} - \{0\}} \exp(-\|m^*/N\|^2)
\]
\[
g(y) \geq e^{-1} \left( 1 + 2 \sum_{[m] \in \mathcal{H} - \{0\}} \exp(-\|m^*/N\|^2) \right), \quad \text{for } \|n_y^*/N\| \leq N.
\]
Therefore (1) follows.
Let $f$ be the trigonometric polynomial such that $f(y) = 1$ for all $y \in E \cap A_2$ and $=0$ otherwise. Let $\alpha = |E \cap A_2|$ and $\delta = \min \{|g(y)| \forall y \in A_2\}$. Then

\[\alpha = \sum_{y \in \Gamma} f(y) \leq \sum_{y \in \Gamma} f(y) g(y) / \delta = (1/\delta) \int_G f(x) g(x) \, dx \leq (1/\delta) \|f\|_q \|g\|_p\]

where $1/p + 1/q = 1$. By the logarithmic convexity of the $\|\cdot\|_q$ norms and the fact that $\|g\|_1 = g(0)$, we have

\[\|g\|_q \leq \|g\|_1^{(2q-1)/q} \|g\|_2^{(2q-1)/q} = g(0)(\|g\|_2/\|g\|_1)^{2/q}.
\]

Therefore $\alpha \leq \|f\|_p (g(0)/\delta)(\|g\|_2/\|g\|_1)^{2/p}$. It follows from the definition of $\Lambda(p)$ set and the estimate (1) that $\alpha \leq B\|f\|_p e(\|g\|_2/\|g\|_1)^{1/q}$ and since $\|f\|_p = \|f\|_p = \alpha^{1/2}$,

$\alpha \leq B^2 e^2(\|g\|_2/\|g\|_1)^{1/p}$. It remains to estimate $\|g\|_2/\|g\|_1$. Generally

\[\|g\|_2 = \frac{(\sum_{m \in \mathbb{H}} (\sum_{m \in \mathbb{H}} \exp(-\|m^*/N\|_2^2))^2)^{1/2}}{\sum_{m \in \mathbb{H}} \exp(-\|m^*/N\|_2^2)}.
\]

When $b_{k+1}, \ldots, b_k$ form an independent subset of $\Gamma$ it is easy to check that $H_0 = 0$. Therefore

\[\|g\|_2/\|g\|_1 = \left( \sum_{m \in \mathbb{H}} \exp(-2\|m^*/N\|_2^2) \right)^{1/2} = ((\beta_1 \cdot \cdot \cdot \beta_k)(1 + (\pi/2)^{1/2}N)^{-k})^{1/2}
\]

and we have the conclusion.

**Remark.** In certain special cases like $s=2$ and $\Gamma=Z$, even when $H_0 \neq 0$ explicit computations show that $\|g\|_2/\|g\|_1 = O(N/s^2)$. However, we are unable to establish this for the general case.

**Theorem 2.** Let $b_1, \ldots, b_k \in \Gamma$ have finite orders $\beta_1, \ldots, \beta_k$ and $b_{k+1}, \ldots, b_s \in \Gamma$ have infinite order. Let $E \subseteq \Gamma$ be a $\Lambda(p)$ set $B$ as in the definition of $\Lambda(p)$ set. If $1 \leq r < \infty$ then

\[|A_r(N, s, b) \cap E| \leq B^2 e^{2\beta_1 \cdot \cdot \cdot \beta_k}2^{s/2}(1 + N^r)^{(s-k)/p}.
\]

**Proof.** Again without loss of generality we take $b_0 = 0$. Consider first the case $r=1$. Define $g_{b_1}, \ldots, g_{b_k}$ as in the proof of Theorem 1, and for $j=k+1, \ldots, s$ define

\[g_{b_j}(x) = \sum_{n \in \mathbb{Z}} \exp(-\|n/N\|_1)(x, nb_j).
\]
Let $g(x) = \prod_{j=1}^{\gamma} g_{b_j}(x)$ then $\hat{g}(\gamma)/\hat{g}(0) > e$ for all $\gamma \in A_1$ and, therefore, $|A_1 \cap E| \leq B_2 \epsilon^2 (\|g\|_2/\|g\|_1)^{4/p}$. Let $n_\gamma$ be any one element in the coset $H_\gamma$; then as before
\[
\|g\|_2 = \left( \sum_{\text{cosets } H_\gamma} \left( \sum_{m \in H_0} \exp(-\|m^* + n_\gamma^*\|/N_1) \right) \right)^{1/2}
\leq \left( \sum_{\text{cosets } H_\gamma} \exp(-2\|n_\gamma^*\|/N_1) \left( \sum_{m \in H_0} \exp(-\|m^*\|/N_1) \right) \right)^{1/2}.
\]
Since $\|g\|_1 = \hat{g}(0) = \sum_{m \in H_0} \exp(-\|m^*\|/N_1)$ we have
\[
\|g\|_2/\|g\|_1 \leq \left( \sum_{\text{cosets } H_\gamma} \exp(-2\|n_\gamma^*\|/N_1) \right)^{1/2} \leq \left( \sum_{m \in H_0} \exp(-2\|n_\gamma^*\|/N_1) \right)^{1/2} ((\beta_1 \cdots \beta_\ell)(1 + N)^{t-k})^{1/2},
\]
and the conclusion follows for $r=1$. Now suppose $1 < r < \infty$. The trivial inequality $\sum_{s-k+1}^{s} |n_j| \leq \sum_{s-k+1}^{s} |n_j|^r$ shows that $A_r(N, s, b) \subseteq A_1(N^r, s, b)$ and the conclusion follows for the general case.

**Corollary.** Suppose $E \subseteq \Gamma$ is a $\Lambda$ set, then
\[
|A_r \cap E| \leq 2C_2 \epsilon^2 \log((\beta_1 \cdots \beta_\ell)(1 + N)^{t-k})
\]
where $C$ is the constant in the definition of $\Lambda$ set.

**Proof.** In Theorem 2 set $p = 2 \log((\beta_1 \cdots \beta_\ell)(1 + N)^{t-k})$.

Kahane [2, p. 58] has already proven this result for the case where $E$ is a Sidon set, and it is unknown if this condition is sufficient to make a set Sidon. The next result shows that in certain cases it is and therefore gives equality of Sidon and $\Lambda$ sets (cf. M. P. Malliavin-Brameret and P. Malliavin [3]).

**Corollary.** Let $E \subseteq \Gamma$ and suppose that each $\gamma \in \Gamma$ has prime order $\beta$. Then the following are equivalent.

(i) $|A_1 \cap E| \leq K \epsilon$ ($K$ depends only on $E$).
(ii) $E$ is a Sidon set.
(iii) $E$ is a $\Lambda$ set.

**Proof.** We know that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Therefore, we show (i) $\Rightarrow$ (ii).

The following result is due to Horn [1].

Let $V$ be a vector space over some field and $E \subseteq V$. Given a natural number $K$, the following two statements are equivalent.

(a) $E$ is the union of $K$ linearly independent sets.
(b) $|B| \leq K \text{ rank}(B)$ for all finite subsets $B \subseteq E$, where $\text{rank}(B)$ is defined as the number of elements in a maximal linearly independent subset of $B$. 

Γ' is a vector space over $Z(\beta)$, and (i) implies that statement (b) of Horn's theorem is satisfied. Thus $E$ is the union of $K$ linearly independent sets and is, therefore, a Sidon set [4, p. 124].

REFERENCES


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