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ABSTRACT. We prove a theorem concerning the arithmetic structure of $\Lambda(p)$ sets. This generalizes a result of Rudin and yields a new characterization of Sidon sets for certain Abelian groups.

Let $G$ be a compact Abelian group with dual group $\Gamma$. A subset $E \subseteq \Gamma$ is called a $\Lambda(p)$ set ($2 < p < \infty$) if for every trigonometric polynomial $f = \sum a_k \gamma_k$ ($\gamma_1, \cdots, \gamma_n \in E$) an inequality $\|f\|_p \leq B \|f\|_2$ holds, where the constant $B$ depends only on $E$ and $p$. Theorems 1 and 2 of this paper give estimates on the number of elements of $E$ which can lie in a given multidimensional arithmetic progression in $\Gamma$. Theorem 2 is a generalization of Rudin's result [5, Theorem 3.5]. When a set $E$ is $\Lambda(p)$ for all $2 < p < \infty$ and the constant $B$ may be taken as $Cp^{1/2}$ where $C$ depends only on $E$ then it is called a $\Lambda$ set. A subset $E \subseteq \Gamma$ is said to be a Sidon set if every bounded function on $E$ can be extended to be a Fourier-Stieltjes transform. It is known that every Sidon set is a $\Lambda$ set, but the converse is unknown [4, p. 128]. As a corollary to Theorem 2 we find that $\Lambda$ sets are also Sidon sets in groups $\Gamma$ where each element has the same prime order.

NOTATION. Suppose $s$ and $N$ are positive integers, $1 \leq r < \infty$, $b_1, \cdots, b_k \in \Gamma$ have finite order, $b_{k+1}, \cdots, b_s \in \Gamma$ have infinite order and $b_0 \in \Gamma$ has arbitrary order. Set

$$A_s(N, s, b) = \left\{n \cdot b + b_0 \mid \sum_{k+1}^{s} |n_j|^r \leq N^r \right\}$$

where $n \cdot b = n_1 b_1 + \cdots + n_s b_s$. Denote by $\hat{f}$ the Fourier transform of $f \in L^1(G)$, by $(x, \gamma)$ the action of the character $\gamma$ on $x \in G$, and by $|F|$ the cardinality of a finite set $F$.

THEOREM 1. Let $b_1, \cdots, b_k \in \Gamma$ have finite orders $\beta_1, \cdots, \beta_k$, and let $b_{k+1}, \cdots, b_s \in \Gamma$ have infinite order. Let $E \subseteq \Gamma$ be a $\Lambda(p)$ set with $B$ a
constant as in the definition of \( \Lambda(p) \) set. If \( A_2(N, s, b) = A_2 \) is as defined above and \( b_{k+1}, \ldots, b_s \) are independent in \( \Gamma \) then

\[
|A_2 \cap E| \leq B^2e^2(\beta_1 \cdots \beta_s)^{2/p}(1 + N(\pi/2)^1/p(s-k)/p).
\]

PROOF. Since translates of \( \Lambda(p) \) sets are \( \Lambda(p) \) sets with the same constant of definition, we may take \( b_0 = 0 \). Define

\[
g_{b_j}(x) = \sum_{n=0}^{\beta_j-1} \langle x, nb_j \rangle \quad \text{for } j = 1, \ldots, k,
\]

\[
g_{b_j}(x) = \sum_{n \in \mathbb{Z}} \exp(-n^2/N^2) \langle x, nb_j \rangle \quad \text{for } j = k + 1, \ldots, s.
\]

It is easy to check that \( g_{b_j}(x) \geq 0 \) for all \( x \in G \).

Let \( \Gamma' = \mathbb{Z}(\beta_1) \oplus \cdots \oplus \mathbb{Z}(\beta_s) \oplus \mathbb{Z}^{s-k} \), then

\[
g(x) = \sum_{n \in \Gamma'} \exp(-\|n^*/N\|_2^2)(x, n \cdot b)
\]

where \( n^* = (0, \ldots, 0, n_{k+1}, \ldots, n_s) \). The map \( \phi(n) = n \cdot b \) is a homomorphism of \( \Gamma' \) into \( \Gamma \). If \( H_\gamma = \phi^{-1}(\gamma) \) then

\[
g(\gamma) = \sum_{n \in H_\gamma} \exp(-\|n^*/N\|_2^2)
\]

when \( \gamma \in \text{Im } \phi \) and \( g(\gamma) = 0 \) otherwise.

We will show that

\[
(1) \quad g(0) \leq e\hat{g}(\gamma) \quad \text{for all } \gamma \in A_2(N, s, b).
\]

Define an equivalence relation on \( H_0 \) by \( h_1 \) is related to \( h_2 \) if \( h_1 = \pm h_2 \). Denote the quotient space by \( \mathcal{H} \) and the class containing \( h \) by \([h]\). Let \( \gamma \in \text{Im } \phi \) be fixed and take \( n_\gamma \in H_\gamma \), then since \( H_\gamma = n_\gamma + H_0 \) we have

\[
g(\gamma) = \sum_{m \in H_0} \exp(-\|(m^* + n_\gamma^*)/N\|_2^2)
\]

\[
= \sum_{[m] \in \mathcal{H}} \sum_{m \in [m]} \exp(-\|m^*/N\|_2^2 - \|n_\gamma^*/N\|_2^2 - 2m^* \cdot n_\gamma^*/N^2)
\]

\[
= \exp(-\|n_\gamma^*/N\|_2^2) \left( 1 + 2 \sum_{[m] \in \mathcal{H} - [0]} \exp(-\|m^*/N\|_2^2) \cosh(2m^* \cdot n_\gamma^*/N^2) \right).
\]

Hence

\[
g(0) = 1 + 2 \sum_{[m] \in \mathcal{H} - [0]} \exp(-\|m^*/N\|_2^2)
\]

\[
\hat{g}(\gamma) \geq e^{-1} \left( 1 + 2 \sum_{[m] \in \mathcal{H} - [0]} \exp(-\|m^*/N\|_2^2) \right), \quad \text{for } \|n_\gamma^*/2 \leq N.
\]

Therefore (1) follows.
Let \( f \) be the trigonometric polynomial such that \( \hat{f}(\gamma) = 1 \) for all \( \gamma \in E \cap A_2 \) and \( =0 \) otherwise. Let \( \alpha = |E \cap A_2| \) and \( \delta = \min \{ \hat{g}(\gamma) | \gamma \in A_2 \} \). Then

\[
\alpha = \sum_{\gamma \in \Gamma} f(\gamma) \\
\leq \sum_{\gamma \in \Gamma} \hat{f}(\gamma) \hat{g}(\gamma) / \delta = (1/\delta) \int_G f(x) g(x) \, dx \\
\leq (1/\delta) \| f \|_0 \| g \|_{q,p}
\]

where \( 1/p + 1/q = 1 \). By the logarithmic convexity of the \( \| \|_q \) norms and the fact that \( \| g \|_1 = \hat{g}(0) \), we have

\[
\| g \|_q \leq \| g \|_1^{(2q-1)/q} \| g \|_2^{(2q-1)/q} = \hat{g}(0) (\| g \|_2 / \| g \|_1)^{2/q}.
\]

Therefore \( \alpha \leq \| f \|_p (\hat{g}(0)/\delta)(\| g \|_2 / \| g \|_1)^{2/p} \). It follows from the definition of \( \Lambda(p) \) set and the estimate (1) that \( \alpha \leq B |f|_p e(\| g \|_2 / \| g \|_1)^{2/p} \) and since \( \| f \|_2 = \| f \|_2 = \alpha^{1/2} \).

\[
\alpha \leq B^2 e^2 (\| g \|_2 / \| g \|_1)^{4/p}.
\]

It remains to estimate \( \| g \|_2 / \| g \|_1 \). Generally

\[
\| g \|_2 / \| g \|_1 = \left( \sum_{m \in H} \exp(-\| m^* / N \|_2^2) \right)^{1/2} / \sum_{m \in H} \exp(-\| m^*/N \|_2^2). \]

When \( b_{k+1}, \ldots , b_k \) form an independent subset of \( \Gamma \) it is easy to check that \( H_0 = 0 \). Therefore

\[
\| g \|_2 / \| g \|_1 = \left( \sum_{n \in \mathbb{Z}} \exp(-2 \| m^* / N \|_2^2) \right)^{1/2} = ((\beta_1 \cdot \cdot \cdot \beta_k)(1 + (\pi/2)^{1/2} N)^{-k})^{1/2}
\]

and we have the conclusion.

**Remark.** In certain special cases like \( s=2 \) and \( \Gamma = \mathbb{Z} \), even when \( H_0 \neq 0 \) explicit computations show that \( \| g \|_2 / \| g \|_1 = O(N^{s/2}) \). However, we are unable to establish this for the general case.

**Theorem 2.** Let \( b_1, \ldots , b_k \in \Gamma \) have finite orders \( \beta_1, \ldots , \beta_k \) and \( b_{k+1}, \ldots , b_s \in \Gamma \) have infinite order. Let \( E \subset \Gamma \) be a \( \Lambda(p) \) set \( B \) as in the definition of \( \Lambda(p) \) set. If \( 1 \leq r < \infty \) then

\[
| A_\infty(N, s, b) \cap E| \leq B^2 e^{2r} (\beta_1 \cdot \cdot \cdot \beta_k)^2 (1 + N^{(r-1)/p}).
\]

**Proof.** Again without loss of generality we take \( b_0 = 0 \). Consider first the case \( r = 1 \). Define \( g_{b_1}, \ldots , g_{b_k} \) as in the proof of Theorem 1, and for \( j = k + 1, \ldots , s \) define

\[
g_{b_j}(x) = \sum_{n \in \mathbb{Z}} \exp(-\| n / N \|_1)(x, nb_j).
\]
Let $g(x) = \prod_{j=1}^{s} g_{y_j}(x)$ then $\hat{g}(\gamma)/\hat{g}(0) > \epsilon$ for all $\gamma \in A_1$ and, therefore, $|A_1 \cap E| \leq B\epsilon^2(\|g\|_2/\|g\|_1)^{1/q}$. Let $n_\gamma$ be any one element in the coset $H_\gamma$; then as before

$$\|g\|_2 = \left( \sum_{\text{cosets } H_\gamma} \left( \sum_{m \in H_0} \exp(-\|m^* + n_\gamma^*\|_1/\|n_\gamma\|_1) \right) \right)^{1/2} \leq \left( \sum_{\text{cosets } H_\gamma} \exp(-2\|n_\gamma^*\|_1) \left( \sum_{m \in H_0} \exp(-\|m^*\|_1) \right) \right)^{1/2}.$$

Since $\|g\|_1 = \hat{g}(0) = \sum_{m \in H_0} \exp(-\|m^*\|_1)$ we have

$$\|g\|_2/\|g\|_1 \leq \left( \sum_{\text{cosets } H_\gamma} \exp(-2\|n_\gamma^*\|_1) \right)^{1/2} \leq \left( \sum_{\text{cosets } H_\gamma} \exp(-2\|n_\gamma^*\|_1) \right)^{1/2} \left( (\beta_1 \cdots \beta_n)(1 + N)^{s-k}\right)^{1/2},$$

and the conclusion follows for $r=1$. Now suppose $1 < r < \infty$. The trivial inequality $\sum_{j=1}^{s} |n_j| \leq \sum_{j=k+1}^{s} |n_j|$ shows that $A_r(N, s, b) \subset A_1(N^r, s, b)$ and the conclusion follows for the general case.

**Corollary.** Suppose $E \subset \Gamma$ is a $\Lambda$ set, then

$$|A_r \cap E| \leq 2C^2\epsilon^3 \log((\beta_1 \cdots \beta_n)(1 + N)^{s-k})$$

where $C$ is the constant in the definition of $\Lambda$ set.

**Proof.** In Theorem 2 set $p = 2 \log((\beta_1 \cdots \beta_n)(1 + N)^{s-k})$.

Kahane [2, p. 58] has already proven this result for the case where $E$ is a Sidon set, and it is unknown if this condition is sufficient to make a set Sidon. The next result shows that in certain cases it is and therefore gives equality of Sidon and $\Lambda$ sets (cf. M. P. Malliavin-Brameret and P. Malliavin [3]).

**Corollary.** Let $E \subset \Gamma$ and suppose that each $\gamma \in \Gamma$ has prime order $\beta$. Then the following are equivalent.

(i) $|A_1 \cap E| \leq K \delta$ ($K$ depends only on $E$).
(ii) $E$ is a Sidon set.
(iii) $E$ is a $\Lambda$ set.

**Proof.** We know that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Therefore, we show (i) $\Rightarrow$ (ii). The following result is due to Horn [1].

Let $V$ be a vector space over some field and $E \subset V$. Given a natural number $K$, the following two statements are equivalent.

(a) $E$ is the union of $K$ linearly independent sets.
(b) $|B| \leq K \text{ rank}(B)$ for all finite subsets $B \subset E$, where $\text{rank}(B)$ is defined as the number of elements in a maximal linearly independent subset of $B$. 

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Γ is a vector space over $Z(\beta)$, and (i) implies that statement (b) of Horn's theorem is satisfied. Thus $E$ is the union of $K$ linearly independent sets and is, therefore, a Sidon set [4, p. 124].

REFERENCES


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