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ARITHMETIC STRUCTURE AND LACUNARY FOURIER SERIES

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ABSTRACT. We prove a theorem concerning the arithmetic structure of $\Lambda(p)$ sets. This generalizes a result of Rudin and yields a new characterization of Sidon sets for certain Abelian groups.

Let G be a compact Abelian group with dual group Γ. A subset $E\subset\Gamma$ is called a $\Lambda(p)$ set $(2<p<\infty)$ if for every trigonometric polynomial $f=\sum_1^n a_k\gamma_k$ $(\gamma_1,\cdots,\gamma_n\in E)$ an inequality $\|f\|_p\leqq B\|f\|_2$ holds, where the constant B depends only on E and p. Theorems 1 and 2 of this paper give estimates on the number of elements of E which can lie in a given multi-dimensional arithmetic progression in Γ. Theorem 2 is a generalization of Rudin's result [**5**, Theorem 3.5]. When a set E is $\Lambda(p)$ for all $2<p<\infty$ and the constant B may be taken as $Cp^{1/2}$ where C depends only on E then it is called a Λ set. A subset $E\subset\Gamma$ is said to be a Sidon set if every bounded function on E can be extended to be a Fourier-Stieltjes transform. It is known that every Sidon set is a Λ set, but the converse is unknown [**4**, p. 128]. As a corollary to Theorem 2 we find that Λ sets are also Sidon sets in groups Γ where each element has the same prime order.

NOTATION. Suppose s and N are positive integers, $1\leqq r<\infty$, $b_1,\cdots,b_k\in\Gamma$ have finite order, $b_{k+1},\cdots,b_s\in\Gamma$ have infinite order and $b_0\in\Gamma$ has arbitrary order. Set

$$A_r(N,s,\boldsymbol{b})=\left\{\boldsymbol{n}\cdot\boldsymbol{b}+b_0 \;\middle|\; \sum_{k+1}^{s}|n_j|^r\leqq N^r\right\}$$

where $\boldsymbol{n}\cdot\boldsymbol{b}=n_1b_1+\cdots+n_sb_s$. Denote by $\hat{f}$ the Fourier transform of $f\in L^1(G)$, by $\langle x,\gamma\rangle$ the action of the character γ on $x\in G$, and by $|F|$ the cardinality of a finite set F.

THEOREM 1. *Let $b_1,\cdots,b_k\in\Gamma$ have finite orders $\beta_1,\cdots,\beta_k$, and let $b_{k+1},\cdots,b_s\in\Gamma$ have infinite order. Let $E\subset\Gamma$ be a $\Lambda(p)$ set with B a*

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constant as in the definition of $\Lambda(p)$ *set. If* $A_2(N, s, \boldsymbol{b})=A_2$ *is as defined above and* $b_{k+1}, \cdots, b_s$ *are independent in* Γ *then*

$$|A_2 \cap E| \leqq B^2 e^2 (\beta_1 \cdots \beta_k)^{2/p} (1 + N(\pi/2)^{1/2})^{2(s-k)/p}.$$

PROOF. Since translates of $\Lambda(p)$ sets are $\Lambda(p)$ sets with the same constant of definition, we may take $b_0=0$. Define

$$g_{b_j}(x) = \sum_{n=0}^{\beta_j - 1} \langle x, nb_j \rangle \qquad \text{for } j = 1, \cdots, k,$$

$$g_{b_j}(x) = \sum_{n \in Z} \exp(-n^2/N^2) \langle x, nb_j \rangle \quad \text{for } j = k+1, \cdots, s.$$

It is easy to check that $g_{b_j}(x) \geqq 0$ for all $x \in G$.

Let $\Gamma' = Z(\beta_1) \oplus \cdots \oplus Z(\beta_k) \oplus Z^{s-k}$, then

$$g(x) = \sum_{\boldsymbol{n} \in \Gamma'} \exp(-\|\boldsymbol{n}^*/N\|_2^2) \langle x, \boldsymbol{n} \cdot \boldsymbol{b} \rangle$$

where $\boldsymbol{n}^* = (0, \cdots, 0, n_{k+1}, \cdots, n_s)$. The map $\phi(\boldsymbol{n}) = \boldsymbol{n} \cdot \boldsymbol{b}$ is a homomorphism of Γ' into Γ. If $H_\gamma = \phi^{-1}(\gamma)$ then

$$\hat{g}(\gamma) = \sum_{\boldsymbol{n} \in H_\gamma} \exp(-\|\boldsymbol{n}^*/N\|_2^2)$$

when $\gamma \in \operatorname{Im} \phi$ and $\hat{g}(\gamma) = 0$ otherwise.

We will show that

$$\hat{g}(0) \leqq e \hat{g}(\gamma) \quad \text{for all } \gamma \in A_2(N, s, \boldsymbol{b}). \tag{1}$$

Define an equivalence relation on H_0 by h_1 is related to h_2 if $h_1 = \pm h_2$. Denote the quotient space by $\mathscr{H}$ and the class containing h by $[h]$. Let $\gamma \in \operatorname{Im} \phi$ be fixed and take $\boldsymbol{n}_\gamma \in H_\gamma$, then since $H_\gamma = \boldsymbol{n}_\gamma + H_0$ we have

$$\begin{aligned} \hat{g}(\gamma) &= \sum_{\boldsymbol{m} \in H_0} \exp(-\|(\boldsymbol{m}^* + \boldsymbol{n}_\gamma^*)/N\|_2^2) \\ &= \sum_{[\boldsymbol{m}] \in \mathscr{H}} \sum_{\boldsymbol{m} \in [\boldsymbol{m}]} \exp(-\|\boldsymbol{m}^*/N\|_2^2 - \|\boldsymbol{n}_\gamma^*/N\|_2^2 - 2\boldsymbol{m}^* \cdot \boldsymbol{n}^*/N^2) \\ &= \exp(-\|\boldsymbol{n}_\gamma^*/N\|_2^2) \left(1 + 2 \sum_{[\boldsymbol{m}] \in \mathscr{H} - [0]} \exp(-\|\boldsymbol{m}^*/N\|_2^2) \cosh(2\boldsymbol{m}^* \cdot \boldsymbol{n}_\gamma^*/N^2)\right). \end{aligned}$$

Hence

$$\hat{g}(0) = 1 + 2 \sum_{[\boldsymbol{m}] \in \mathscr{H} - [0]} \exp(-\|\boldsymbol{m}^*/N\|_2^2)$$

$$\hat{g}(\gamma) \geqq e^{-1} \left(1 + 2 \sum_{[\boldsymbol{m}] \in \mathscr{H} - [0]} \exp(-\|\boldsymbol{m}^*/N\|_2^2)\right), \quad \text{for } \|\boldsymbol{n}_\gamma^*\|_2 \leqq N.$$

Therefore (1) follows.

Let f be the trigonometric polynomial such that $\hat{f}(\gamma)=1$ for all $\gamma \in E\cap A_2$ and $=0$ otherwise. Let $\alpha=|E\cap A_2|$ and $\delta=\min\{\hat{g}(\gamma)|\gamma \in A_2\}$. Then

$$\begin{aligned}\alpha &= \sum_{\gamma\in\Gamma}\hat{f}(\gamma)\\ &\leqq \sum_{\gamma\in\Gamma}\hat{f}(\gamma)\hat{g}(\gamma)/\delta = (1/\delta)\int_G f(x)g(x)\,dx\\ &\leqq (1/\delta)\|f\|\ g\|_{qp}\end{aligned}$$

where $1/p+1/q=1$. By the logarithmic convexity of the $\|\ \|_q$ norms and the fact that $\|g\|_1=\hat{g}(0)$, we have

$$\|g\|_q \leqq \|g\|_1^{(2-q)/q}\,\|g\|_2^{(2q-2)/q} = \hat{g}(0)(\|g\|_2/\|g\|_1)^{2/p}.$$

Therefore $\alpha\leqq\|f\|_p\,(\hat{g}(0)/\delta)(\|g\|_2/\|g\|_1)^{2/p}$. It follows from the definition of $\Lambda(p)$ set and the estimate (1) that $\alpha\leqq B\|f\|_2\,e(\|g\|_2/\|g\|_1)^{2/p}$ and since $\|f\|_2=\|\hat{f}\|_2=\alpha^{1/2}$,

$$\alpha \leqq B^2e^2(\|g\|_2/\|g\|_1)^{4/p}.$$

It remains to estimate $\|g\|_2/\|g\|_1$. Generally

$$\frac{\|g\|_2}{\|g\|_1}=\frac{(\sum_{\text{cosets } H\gamma}(\sum_{m\in H\gamma}\exp(-\|\boldsymbol{m}^*/N\|_2^2))^2)^{1/2}}{\sum_{m\in H_0}\exp(-\|\boldsymbol{m}^*/N\|_2^2)}.$$

When $b_{k+1},\cdots,b_k$ form an independent subset of Γ it is easy to check that $H_0=0$. Therefore

$$\|g\|_2/\|g\|_1 = \left(\sum_{n\in\Gamma'}\exp(-2\,\|\boldsymbol{m}^*/N\|_2^2)\right)^{1/2} = ((\beta_1\cdots\beta_k)(1+(\pi/2)^{1/2}N)^{s-k})^{1/2}$$

and we have the conclusion.

REMARK. In certain special cases like $s=2$ and $\Gamma=Z$, even when $H_0\neq 0$ explicit computations show that $\|g\|_2/\|g\|_1=O(N^{s/2})$. However, we are unable to establish this for the general case.

THEOREM 2. *Let $b_1,\cdots,b_k\in\Gamma$ have finite orders $\beta_1,\cdots,\beta_k$ and $b_{k+1},\cdots,b_s\in\Gamma$ have infinite order. Let $E\subset\Gamma$ be a $\Lambda(p)$ set B as in the definition of $\Lambda(p)$ set. If $1\leqq r<\infty$ then*

$$|A_r(N,s,b)\cap E| \leqq B^2e^2(\beta_1\cdots\beta_k)^{2/p}(1+N^r)^{2(s-k)/p}.$$

PROOF. Again without loss of generality we take $b_0=0$. Consider first the case $r=1$. Define $g_{b_1},\cdots,g_{b_k}$ as in the proof of Theorem 1, and for $j=k+1,\cdots,s$ define

$$g_{b_j}(x) = \sum_{n\in Z}\exp(-\|n/N\|_1)\langle x, nb_j\rangle.$$

Let $g(x)=\prod_{j=1}^{s} g_{b_j}(x)$ then $\hat{g}(\gamma)/\hat{g}(0)>e$ for all $\gamma\in A_1$ and, therefore, $|A_1\cap E|\leqq B^2e^2(\|g\|_2/\|g\|_1)^{4/p}$. Let $\boldsymbol{n}_\gamma$ be any one element in the coset H_γ; then as before

$$\|g\|_2=\left(\sum_{\text{cosets } H_\gamma}\left(\sum_{\boldsymbol{m}\in H_0}\exp(-\|(\boldsymbol{m}^*+\boldsymbol{n}_\gamma^*)/N\|_1)\right)^2\right)^{1/2}$$
$$\leqq\left(\sum_{\text{cosets } H_\gamma}\exp(-2\,\|\boldsymbol{n}_\gamma^*/N\|_1)\left(\sum_{\boldsymbol{m}\in H_0}\exp(-\|\boldsymbol{m}^*/N\|_1)\right)^2\right)^{1/2}.$$

Since $\|g\|_1=\hat{g}(0)=\sum_{\boldsymbol{m}\in H_0}\exp(-\|\boldsymbol{m}^*/N\|_1)$ we have

$$\|g\|_2/\|g\|_1\leqq\left(\sum_{\text{cosets } H_\gamma}\exp(-2\,\|\boldsymbol{n}_\gamma^*/N\|_1)\right)^{1/2}$$
$$\leqq\left(\sum_{\boldsymbol{n}\in\Gamma'}\exp(-2\,\|\boldsymbol{n}^*/N\|_1)\right)^{1/2}((\beta_1\cdots\beta_k)(1+N)^{s-k})^{1/2},$$

and the conclusion follows for $r=1$. Now suppose $1<r<\infty$. The trivial inequality $\sum_{j=k+1}^{s}|n_j|\leqq\sum_{j=k+1}^{s}|n_j|^r$ shows that $A_r(N,s,\boldsymbol{b})\subset A_1(N^r,s,\boldsymbol{b})$ and the conclusion follows for the general case.

COROLLARY. *Suppose $E\subset\Gamma$ is a Λ set, then*

$$|A_r\cap E|\leqq 2C^2e^3\log((\beta_1\cdots\beta_k)(1+N^r)^{s-k})$$

where C is the constant in the definition of Λ set.

PROOF. In Theorem 2 set $p=2\log((\beta_1\cdots\beta_k)(1+N^r)^{s-k})$.

Kahane [**2**, p. 58] has already proven this result for the case where E is a Sidon set, and it is unknown if this condition is sufficient to make a set Sidon. The next result shows that in certain cases it is and therefore gives equality of Sidon and Λ sets (cf. M. P. Malliavin-Brameret and P. Malliavin [**3**]).

COROLLARY. *Let $E\subset\Gamma$ and suppose that each $\gamma\in\Gamma$ has prime order β. Then the following are equivalent.*

(i) $|A_1\cap E|\leqq Ks$ (*K depends only on E*).

(ii) *E is a Sidon set.*

(iii) *E is a Λ set.*

PROOF. We know that (ii)$\Rightarrow$(iii)$\Rightarrow$(i). Therefore, we show (i)$\Rightarrow$(ii). The following result is due to Horn [**1**].

Let V be a vector space over some field and $E\subset V$. Given a natural number K, the following two statements are equivalent.

(a) E is the union of K linearly independent sets.

(b) $|B|\leqq K\,\text{rank}(B)$ for all finite subsets $B\subset E$, where rank(B) is defined as the number of elements in a maximal linearly independent subset of B.

Γ is a vector space over $Z(\beta)$, and (i) implies that statement (b) of Horn's theorem is satisfied. Thus E is the union of K linearly independent sets and is, therefore, a Sidon set [**4**, p. 124].

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