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ON CERTAIN FIBERINGS OF $M^2 \times S^1$

WOLFGANG HEIL[1]

ABSTRACT. Using a theorem of Stallings it is shown that the product of S^1 and a surface of genus $g>1$ admits for every integer $n \geqq 0$ a fibering over S^1 with a surface of genus $n(g-1)+g$ as fiber. Conversely, these are all possible such fibrations (up to equivalence). Let N be a Seifert fiber space which is locally trivial fibered over S^1 with fiber a surface. It is shown that any two such fiberings of N over S^1 are equivalent if the fibers are homeomorphic.

In [8] and [1] it is shown that the 3-manifold $M=F\times S^1$, where F is an orientable closed surface of genus $g>1$, admits for every number $n\geqq 0$ a fibering over S^1 with a surface T_n of genus $n(g-1)+g$ as fiber. In this note we show that this result follows immediately from Stallings' theorem [7] (this applies also if F is bounded or nonorientable). It is shown that these are all possible fibrations of M over S^1 with fiber a surface and this is generalized to Seifert fiber spaces.

1. Let F be an orientable surface of genus $g>1$ and m boundary components, let $M=F\times S^1$, $\mathfrak{G}=\pi_1(M)$,

$$\mathfrak{G} = \{a_1, b_1, \cdots, a_g, b_g, s_1, \cdots, s_m, h : s_1 \cdots s_m[a_1, b_1] \cdots [a_g, b_g] = 1,$$
$$[a_i, h] = [b_i, h] = [s_k, h] = 1 \ (i = 1, \cdots, g; k = 1, \cdots, m)\}.$$

Let Z be represented by the group of integers and construct an epimorphism $\phi : \mathfrak{G} \to Z$ as follows

$$\begin{aligned} \phi(a_1) &= 1, \\ \phi(a_i) &= \phi(b_j) = 0 \qquad (i = 2, \cdots, g; j = 1, \cdots, g), \\ \phi(h) &= n > 0, \\ \phi(s_k) &= \gamma_k \qquad (k = 1, \cdots, m). \end{aligned}$$

(γ_k are arbitrary integers, subject to the condition $\gamma_1+\cdots+\gamma_m=0$.)

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(a) If F is closed (i.e. $m=0$), computing $\mathfrak{N}_n=\ker \phi$ using the Reidemeister-Schreier method, we obtain

$$\mathfrak{N}_n = \Big\{a_{i,k}, b_{j,k}, h_k\colon a_{i,k}h_k a_{i,k+n}^{-1}h_k^{-1} = 1,\ b_{j,k}h_k b_{j,k+n}^{-1}h_k^{-1} = 1,$$

$$h_{k+1}h_k^{-1} = 1,\ b_{1,k+1}b_{1,k}^{-1}\prod_{l=2}^{g}[a_{l,k}, b_{l,k}] = 1$$

$$(i = 2, \cdots, g; j = 1, \cdots, g; -\infty < k < \infty)\Big\}.$$

Here $a_{i,k}=a_1^k a_i a_1^{-k}$, $b_{j,k}=a_1^k b_j a_1^{-k}$, $h_k=a_1^k h a_1^{-(k+n)}$. This is equivalent to

$$\mathfrak{N}_n = \Big\{h_0, b_{1,1}, a_{i,1}, b_{i,1}, \cdots, a_{i,n}, b_{i,n}\colon [h_0^{-1}, b_{1,1}]\prod_{j=2}^{g}[a_{j,1}, b_{j,1}]$$

$$\times \prod_{j=2}^{g}[a_{j,2}, b_{j,2}]\cdots\prod_{j=2}^{g}[a_{j,n}, b_{j,n}] = 1\ (i = 2, \cdots, g)\Big\}$$

which is the fundamental group of an orientable closed surface of genus $n(g-1)+1$. Thus the theorem in the introduction follows by applying Stallings' theorem [7].

(b) If $\partial M \neq \varnothing$ (i.e. $m>0$) we obtain, for $\mathfrak{N}_n=\ker \phi$,

$$\mathfrak{N}_n = \{a_{i,k}, b_{j,k}, s_{l,k}, ha_1^{-n} (i = 2, \cdots, g; j = 1, \cdots, g;$$
$$k = 0, \cdots, n-1; l = 1, \cdots, m-1)\}$$
$$(\text{where } s_{l,k} = a_1^k s_l a_1^{\gamma_l - k}),$$

a free group of rank $n(2g+m-2)+1$. By Stallings' theorem M fibers over S^1 with fiber a surface T_n with $\pi_1(T_n)=\mathfrak{N}_n$. M is a (trivial) Seifert fiber space with orbit surface F. T_n is a branched covering of F (see the proposition, §3). Since M has no singular fibers this covering is without branch points. Thus if g' denotes the genus and m' the number of boundary components of T_n and if the covering $T_n \to F$ is η-sheeted, we have for the Euler characteristics

$$2g' + m' - 2 = \eta(2g + m - 2) = n(2g + m - 2).$$

Thus: *For every natural number n there exists a surface T_n which is an n-sheeted covering of F and such that M admits a fibering over S^1 with fiber T_n.*

(c) The same method carries over to the nonorientable case.

2. The fiberings of §1 are all possible fiberings of M over S^1 with fiber a surface. This can be seen as follows:

Let $\phi\colon \mathfrak{G} \to \boldsymbol{Z}$ be any epimorphism.

Let

$$\begin{aligned} \phi(a_i) &= \alpha_i \qquad (i = 1, \cdots, g), \\ \phi(b_i) &= \beta_i \qquad (i = 1, \cdots, g), \\ \phi(s_k) &= \gamma_k \qquad (k = 1, \cdots, m), \\ \phi(h) &= n. \end{aligned}$$

Let g.c.d.$(\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g)=d$. Since ϕ is an epimorphism, we have g.c.d.$(d, \gamma_1, \cdots, \gamma_m, n)=1$.

The assertion follows from the following:

LEMMA. *Let $\phi: \mathfrak{G} \to \boldsymbol{Z}$ be any epimorphism and let x be any one of the generators $a_1, b_1, \cdots, a_g, b_g$. Then there exists an automorphism μ of $\mathfrak{G}$ which is induced by a homeomorphism of M, such that $\phi \cdot \mu(x)=$*g.c.d.(d, n) *and $\phi \cdot \mu(y)=0$, where $y \in \{a_1, b_1, \cdots, a_g, b_g\}-\{x\}$. If F is not a torus, we may assume $\phi(h)>0$.*

PROOF. μ is a composition of the following automorphisms (we write down the generators which are not kept fixed).

$$\begin{aligned} \mu_1^{(i)}(a_i) &= a_i b_i^k \qquad (k \in \mathbf{Z}) \quad (i = 1, \cdots, g), \\ \mu_2^{(i)}(b_i) &= b_i a_i^l \qquad (l \in \mathbf{Z}) \quad (i = 1, \cdots, g), \\ \mu_3(a_1) &= a_1 a_2 b_2^{-1}, \\ \mu_3(b_1) &= b_2 a_2^{-1} b_1 a_2 b_2^{-1}, \\ \mu_3(a_2) &= b_2 a_2^{-1} b_1 a_2 b_2^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_1^{-1} a_2 b_2^{-1}, \\ \mu_3(b_2) &= b_2 b_2 a_2^{-1} b_1^{-1} a_2^{-1} b_2^{-1}, \\ \mu_4(a_1) &= a_1 a_2^{-1} b_2^{-1}, \\ \mu_4(b_1) &= b_2 a_2 b_1 a_2^{-1} b_2^{-1}, \\ \mu_4(a_2) &= b_2 a_2 b_1 a_2^{-1} b_2^{-1} b_1^{-1} b_2^{-1} b_1^{-1} a_2^{-1} b_2^{-1}, \\ \mu_4(b_2) &= b_2 a_2 b_2^{-1} a_2^{-1} b_2^{-1} a_2^{-1}, \\ \mu_5^{(i)}(a_i) &= a_{i+1}, \\ \mu_5^{(i)}(b_i) &= b_{i+1}, \\ \mu_5^{(i)}(a_{i+1}) &= [a_{i+1}, b_{i+1}]^{-1} a_i [a_{i+1}, b_{i+1}], \\ \mu_5^{(i)}(b_{i+1}) &= [a_{i+1}, b_{i+1}]^{-1} b_i [a_{i+1}, b_{i+1}] \qquad (i \text{ taken mod } g), \\ \mu_6(a_1) &= a_1 h^{\pm 1}, \\ \mu_7(h) &= h^{-1}. \end{aligned}$$

It is not difficult to see that these are automorphisms and furthermore that they are induced by homeomorphisms of M, since they leave the peripheral system of $\mathfrak{G}$ fixed (see [**2**]). These automorphisms were suggested by the paper of J. Nielsen [**3**].

Let

$$A = \begin{pmatrix} \alpha_1, & \beta_1 \\ \cdot & \\ \cdot & \\ \cdot & \\ \alpha_g, & \beta_g \end{pmatrix}.$$

The automorphisms μ_1 and μ_2 change the map ϕ as follows:

$$(\mu_1) \qquad \phi(a_i) \to \phi(a_i) + k\phi(b_i),$$

$$(\mu_2) \qquad \phi(b_i) \to \phi(b_i) + l\phi(a_i).$$

Using the Euclidean algorithm and (μ_1), (μ_2), we transform A into

$$A' = \begin{pmatrix} d_1, & 0 \\ \cdot & \\ \cdot & \\ \cdot & \\ d_g, & 0 \end{pmatrix}, \quad \text{where } d_i = (\alpha_i, \beta_i).$$

Similarly, using μ_3, μ_4, μ_5, μ_6 we change A' into

$$\begin{pmatrix} 0, & \text{g.c.d.}(d, n) \\ 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 \end{pmatrix}.$$

The last statement of the lemma follows by considering μ_7 and observing that $(\ker \phi) \cap Z(h) = 1$, where $Z(h)$ is the cyclic subgroup of $\mathfrak{G}$ generated by h (see [**5**, proof of Satz 7]).

3. **A comparison of Nielsen's and Seifert's invariants.** In this section we show how for Seifert fiber spaces that are fibered over S^1 the fiber is a branched covering of the Seifert (orbit) surface. This will be used (in the next section) to show the uniqueness of the fibration as mentioned in the introduction.

Let $\phi: F \to F$ be an orientation preserving homeomorphism of finite order n of a (compact) orientable surface F of genus g and r boundary components. Let P be a fixed point of order λ. The orbitspace of ϕ is a surface $\bar{F}$ and P covers a point $\bar{P} \in \bar{F}$. A simple closed curve s in F which covers a simple closed curve $\bar{s}$ about $\bar{P}$, covers it λ times. We have m

disjoint curves lying over $\bar{s}$, where $n=\lambda\cdot m$. Choose an orientation on F.

Let $\bar{Q}$ be any point on $\bar{s}$. Q is covered by λ points on s lying over $\bar{Q}$. The (oriented) arc on s which starts at Q and covers $\bar{s}$ once ends at a certain point $\phi^{\sigma m}Q$. Note that g.c.d.$(\sigma, \lambda)=1$. The *valenz* of P is defined to be the triple (m, λ, σ). A multiple point is one for which $\lambda>0$.

THEOREM (NIELSEN [3]). *Let F, F' be homeomorphic closed surfaces, let $\phi:F\to F$ and $\phi':F'\to F'$ be homeomorphisms of finite order n. Then ϕ and ϕ' are equivalent (i.e. there exists a homeomorphism $\psi:F\to F'$ such that $\phi\psi=\psi\phi'$) iff F and F' have the same valenz-numbers at multiple points.*

For a description of the Seifert invariants (μ, ν) of a fibered solid torus and a 3-manifold, see [**6**].

Let M be a Seifert fiber space which admits a fibering over S^1 with fiber a surface F of genus >1. Thus M can be obtained from $F\times I$ (where I denotes the unit interval $[0, 1]$) by identifying $F\times 0$ with $\phi F\times 1$, where $\phi:F\to F$ is a homeomorphism and we write $M=F\times I/\phi$. It is easy to see that M is a Seifert fiber space iff $\pi_1(M)$ has nontrivial center and $\pi_1(M)$ has nontrivial center iff ϕ is isotopic to a homeomorphism ϕ' of finite order (see e.g. [**9**, p. 514]). Since ϕ and ϕ' determine homeomorphic 3-manifolds [**2**], we may assume that ϕ has finite order n. We construct a Seifert fibration of M as follows: Let P in F be a fixed point of order $\lambda>1$. Then $P, \phi P, \cdots, \phi^{m-1}(P)$ (where $\lambda m=n$) cover the same point $\bar{P}$ in the orbit surface $\bar{F}$. Now $F\times I$ has a trivial fibering as a line bundle. Take a neighborhood $U(P)$ of P which does not contain any other multiple point and such that $\phi^m(U(P))=U(P)$. Then we have neighborhoods

$$U(P)\times I, \phi U(P)\times I, \cdots, \phi^{m-1}U(P)\times I \qquad (\phi^m U(P)=U(P))$$

of $P\times I, \phi P\times I, \cdots, \phi^{m-1}(P)\times I$ in $F\times I$ and they match together to form a fibered solid torus in M. The fiber in M which contains P is composed of m lines $P\times I, \cdots, \phi^{m-1}(P)\times I$ and the fiber through any point Q of $U(P)$ $(Q\neq P)$ is composed of n lines $Q\times I, \cdots, \phi^{n-1}(Q)\times I$. Hence the fibers through $U(P)$ form a fibered neighborhood of an exceptional fiber of order $n/m=\lambda$.

Note that the orbit surface $\bar{F}$ is the Seifert surface of the Seifert fibration.

Let $\bar{P}\in\bar{F}$ be a multiple point of order λ, $\bar{s}$ a small simple closed curve around $\bar{P}$, $\bar{Q}\in\bar{s}$ an arbitrary point and s in F a closed curve which covers $\bar{s}$. On s there are exactly λ points which cover $\bar{Q}$:

$$Q, \phi^{\sigma m}Q, \cdots, \phi^{(\lambda-1)\sigma m}Q \quad \text{(exponents mod } n),$$

where σ is the valenz. To find $\phi^m Q$ in this sequence, we have to find an integer δ such that $\delta\sigma\equiv 1\ (\lambda)$. Now s is mapped onto itself for the first time by ϕ^m and ϕ^m is equivalent to a rotation of $2\pi\delta/\lambda$ of a circle. Hence the

Seifert invariants μ, ν of M and the valenz (m, λ, σ) of the map $\phi: F \to F$ satisfy

$$\sigma \equiv \nu \pmod{\mu}, \quad \text{where } \delta\sigma \equiv 1 \pmod{\lambda},$$
$$\lambda = \mu.$$

Now if M_1 and M_2 are homeomorphic Seifert fiber spaces, then the corresponding Seifert surfaces are homeomorphic and M_1 and M_2 have the same numbers μ, ν by the classification theorem of Seifert fiber spaces **[5]**.

Hence we have the following:

PROPOSITION. *If $M_1 = F_1 \times I/\phi_1$ and $M_2 = F_2 \times I/\phi_2$ are homeomorphic and ϕ_i is a homeomorphism of order n_i $(i=1, 2)$, then F_1 and F_2 are (branched) coverings of the same orbit surface (=Seifert surface) $\bar{F}$ with the same number t of branch points (on $\bar{F}$) which are of the same orders λ.*

4. **Equivalent Stallings fibrations.** Two fiberings (M_1, p_1, S^1, F_1) and (M_2, p_2, S^1, F_2) are equivalent iff there exists a homeomorphism $\psi: M_1 \to M_2$ with $\psi p_2 = p_1$. Let F_i be a closed orientable surface of genus $g_i > 1$ $(i = 1, 2)$ and let $\phi_i: F_i \to F_i$ be a homeomorphism of finite order n_i.

THEOREM. *Let $M_i = F_i \times I/\phi_i$ $(i=1, 2)$. Assume F_1 and F_2 are homeomorphic. Then the following are equivalent:*

(a) *M_1 is homeomorphic to M_2.*
(b) *M_1 is equivalent to M_2.*
(c) *ϕ_1 is equivalent to ϕ_2 (and is of the same order).*

In particular, it follows that if M is a closed Seifert fiber space which admits two fibrations over S^1 with fibers F_1 and F_2, then either F_1 is not homeomorphic to F_2 or the two fibrations are equivalent.

PROOF. If ϕ_1 and ϕ_2 are equivalent then it is not hard to see that M_1 and M_2 are equivalent (see e.g. **[2]**). Thus (c)→(b)→(a). We show (a)→(c): Let M_1 be homeomorphic to M_2. M_1 and M_2 are Seifert fiber spaces and have the same Seifert surface $\bar{F}$. If t_i denotes the number of branch points (on $\bar{F}$) of the orbit surfaces of ϕ_i $(i=1, 2)$ and $\lambda_j^{(i)}$ the orders of the branch points $(i=1, 2; j=1, \cdots, t_i)$ we have (by the proposition) $t_1 = t_2 = t$ and $\lambda_j^{(1)} = \lambda_j^{(2)} = \lambda_j$ $(j=1, \cdots, t)$. Consider the branched covering $F_i \to \bar{F}$ $(i=1, 2)$ and cut out a small disc D_j in $\bar{F}$ containing a branch point of order $\lambda_j^{(i)}$ and remove the $m_j^{(i)}$ discs in F_i which cover D_j (where $n_i = \lambda_j^{(i)} m_j^{(i)}$). Do this for all branch points $\bar{P}_j$ $(j=1, \cdots, t)$ and get an unbranched covering $F_i' \to \bar{F}'$. Clearly, if r_i denotes the number of boundary components of F_i', we have

$$r_i = m_1^{(i)} + \cdots + m_t^{(i)} \qquad (i = 1, 2).$$

Using this equation together with $n_i = \lambda_j m_j^{(i)}$ $(i=1, 2; j=1, \cdots, t)$ and comparing the Euler characteristics of F_i' and $\bar{F}'$ we get $n_1 = n_2$ and $m_j^{(1)} = m_j^{(2)}$.

Now ϕ_1 and ϕ_2 are of the same orders and have the same valenz-numbers at the fixed points. By the Nielsen equivalence theorem ϕ_1 and ϕ_2 are equivalent.

REMARK. A "mapping class" is a coset of the group of all homeomorphisms of a surface F modulo the subgroup of isotopic deformations. J. Nielsen [**4**, p. 24] proves that a mapping class of order n contains a homeomorphism of order n. The above theorem shows that there is exactly one such homeomorphism (up to equivalence).

For let $\phi: F \to F$ be a homeomorphism of order n and ψ be a homeomorphism of the same class. Then $M = F \times I/\phi \approx F \times I/\psi$. If ψ has finite order, then by the theorem, ψ has order n and is equivalent to ϕ.

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DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306