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INVOLUTORIAL DIVISION RINGS WITH ARBITRARY CENTERS[1]

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ABSTRACT. It is proved that for an arbitrary field k there exists an involutorial division ring having k as its center.

The aim of this note is to prove that for any given commutative field k, there exists an involutorial division ring whose center is k. The problem whether such division rings exist has been raised by Professor Tamagawa for finite fields, and more generally for fields of finite characteristic. If the given field has characteristic 0, one can construct such division rings using differential polynomials [**6**]. Indeed, let $K=k(x)$ be the field of rational functions in one indeterminate x over k, and denote its usual derivation by δ. Let $R=K[t,\delta]$ be the ring of differential polynomials in t with coefficients in K written on the right, and with the product determined by $at=ta+a\delta$, for $a\in K$. The ring R can be identified with a ring of differential operators and the mapping J of R that sends each operator $\sum_{i=0}^{n} t^i a_i$ to its adjoint $\sum_{i=0}^{n}(-1)^i a_i t^i$ is an involution [**1**, p. 248]. Since R is an Ore domain, it has a field of quotients D, and the involution J can be extended (in a unique way) to an involution of D [**8**]. Using well-known properties of the ring of polynomials R, and the fact that k has characteristic 0, it is not hard to show that the center of D is k.

It was surprisingly difficult to get the result for a field k of finite characteristic p. In this case the center of the division ring D, just constructed, is much larger than k. We also obtained centers larger than k, when we constructed other Ore domains with involution, containing k.

This situation led us to the consideration of the free associative algebra $k\langle X\rangle$, on a set X of at least two elements, over k. This is an involutorial ring. Indeed, the free semigroup generated by X has an involution which is defined by mapping each word into its opposite. This map, when extended by linearity to $k\langle X\rangle$, becomes an involution of $k\langle X\rangle$. The ring

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$k\langle X\rangle$ is well known to be embeddable in a division ring and there are several methods of embedding. The method of Mal'cev and Neumann [**2**, p. 276] cannot be used for our purpose. Indeed, the involution of $k\langle X\rangle$ we have defined, can be extended to an involution of the group algebra over k of the free group G on X, but taking any ordering of G, it is easy to see that there exists an increasing sequence which is mapped into a decreasing one by the involution. This implies that the involution cannot be extended to the division ring of power series with well-ordered support. Also, using the method of Jategaonkar [**7**] of embedding $k\langle X\rangle$ into a division ring, one observes that the involution cannot be extended.

In the following we shall prove that any involution of $k\langle X\rangle$ can be extended to the universal field of fractions of $k\langle X\rangle$ defined by P. M. Cohn [**3**] and [**5**]. It will be proved that the center of the universal field of fractions of $k\langle X\rangle$ is k, if X is an infinite set.

Cohn has pointed out in [**3**, p. 613] that any automorphism of a fir R can be extended in a unique way to its universal field of fractions. We shall show that the same is true for an anti-automorphism and by the uniqueness of the extension it will follow that an involution extends to an involution. This result will be applied to $k\langle X\rangle$ which is a fir and has an involution.

We assume as in [**3**] that every ring has a unit element which is preserved by homomorphisms and we shall use the definitions and the results mentioned there.

Given a ring R we denote its opposite by R°. It is assumed that $R^\circ=R$ as additive groups and the product in R° is given by $a\times b=ba$. If $f:R\to S$ is a homomorphism, then so is $f:R^\circ\to S^\circ$ and the map $R\to R^\circ, f\to f$ is an isomorphism of the category of rings onto itself. Hence this map preserves commutative diagrams. In addition we shall use the following remarks:

(i) $J:R\to R$ is an anti-automorphism if and only if either $J:R\to R^\circ$ or $J:R^\circ\to R$ is an isomorphism.

(ii) If A, B, C are $n\times r$, $r\times n$, $n\times n$ matrices respectively, over $R=R^\circ$, then $AB=C$ over R if and only if $B^TA^T=C^T$ over R°, where as usual the transpose of a matrix A is denoted by A^T.

(iii) Given a set M of square matrices over R, we denote the set $\{A^T \mid A\in M\}$ by M^T. If $f:R\to S$ is a homomorphism, then it is M-inverting (i.e. the image of any $A\in M$ is invertible over S) if and only if $f:R^\circ\to S^\circ$ is M^T-inverting.

(iv) For any set M of square matrices over a ring R, there exists a universal M-inverting ring R_M with the corresponding M-inverting homomorphism $\lambda:R\to R_M$ [**3**]. As a result of the universal property of R_M and λ, it follows that if $\tau:R\to S$ is an isomorphism and $N=M^T=\{A^T \mid A\in M\}$, and if S_N and $\nu:S\to S_N$ are the corresponding universal N-inverting ring and homomorphism, then there exists one and only one isomorphism $\tau':R_M\to S_N$ satisfying $\lambda\tau'=\tau\nu$. In particular if $R=S$ and τ is the identity map, then τ' is the identity map on R_M.

By (iii), $\lambda:R^\circ\to(R_M)^\circ$ is M^T-inverting. If $f:R^\circ\to S^\circ$ is M^T-inverting, then $f:R\to S$ is M-inverting, hence there exists a unique homomorphism $f':R_M\to S$ satisfying $f=\lambda f'$. Therefore $f':(R_M)^\circ\to S^\circ$ is the unique homomorphism satisfying $f=\lambda f'$. This proves:

LEMMA 1. *Let $\lambda:R\to R_M$ be the universal M-inverting homomorphism and $N=M^T$. Then $(R_M)^\circ$ is a universal N-inverting ring for R° and $\lambda:R^\circ\to(R_M)^\circ$ is the corresponding universal N-inverting homomorphism.*

Next we have:

LEMMA 2. *If R has an anti-automorphism J and $M^{JT}=M$, then there exists one and only one anti-automorphism J' of R_M satisfying $\lambda J'=J\lambda$. Moreover, if J is an involution then so is J'.*

PROOF. By (i), $J:R\to R^\circ$ is an isomorphism and M is mapped onto $N=M^J$. Since $M^{JT}=M$ it follows that $N=M^T$. By the previous lemma and by (iv), there exists a unique isomorphism $J':R_M\to(R_M)^\circ$ satisfying $\lambda J'=J\lambda$. Hence $J':R_M\to R_M$ is the unique anti-automorphism satisfying $\lambda J'=J\lambda$.

If J is an involution, then $J^2=1$. Consider the commutative diagram:

$$\begin{tikzcd}
R \arrow[r, "J"] \arrow[d, "\lambda"'] & R^\circ \arrow[r, "J"] \arrow[d, "\lambda"'] & R \arrow[d, "\lambda"] \\
R_M \arrow[r, "J'"'] & (R_M)^\circ \arrow[r, "J'"'] & R_M
\end{tikzcd}$$

Since $J^2=1$ and $\lambda J'^2=J^2\lambda=\lambda=\lambda\cdot 1$ it follows (see the end of (iv)) that $J'^2=1$, hence J' is an involution.

Now, let R be a fir and let F be the set of all full matrices over R. If $A\in F$ and J is an anti-automorphism of R, then since J^{-1} is an anti-automorphism, it is readily seen that A^{JT} is full. It follows that $F^{JT}=F$ and as a result of the previous lemma we get:

THEOREM. *Let R be a fir and J an anti-automorphism of R. Then there exists one and only one extension of J to an anti-automorphism J' of the universal field of fractions R_F of R. If J is an involution then so is J'.*

Let us denote the universal field of fractions of $k\langle X\rangle$ by $k(X)$. As it has already been remarked, the theorem applied to $k\langle X\rangle$ implies that $k(X)$ is an involutorial division ring. It remains to prove that the center of $k(X)$

is k. It is clear that the center of $k(X)$ contains k. We have to prove that given $a \in k(X)$, $a \notin k$, there exists $b \in k(X)$ satisfying $ab \neq ba$. We shall prove that if X is infinite then there exists some $x \in X$ satisfying $ax \neq xa$.

First note that by [**3**, Theorem 7.3] and [**4**, Theorem 3.3], if R is a fir and F the set of all full matrices over R, then the universal F-inverting ring R_F is the universal field of fractions of R and it is an honest closure of R.

If Y is a subset of X then we regard $k\langle Y\rangle$ as a subring of $k\langle X\rangle$. If A is a full matrix over $k\langle Y\rangle$, then it is full over $k\langle X\rangle$. This is obtained by applying the homomorphism $k\langle X\rangle \rightarrow k\langle Y\rangle$ that sends each element of Y into itself and the other elements of X into 0. Let F be the set of all full matrices over $k\langle X\rangle$, then $k(X)$ is the universal F-inverting ring. Hence the full matrices over $k\langle Y\rangle$ have inverses over $k(X)$ and the ring generated by the elements of these matrix inverses is an honest closure of $k\langle Y\rangle$, and we may denote it by $k(Y)$. So we have that $Y \subseteq X$ implies $k(Y) \subseteq k(X)$. Moreover, if $Y \subseteq Y' \subseteq X$, it is clear that $k(Y) \subseteq k(Y')$.

Since $k(X)$ is the universal field of fractions of $k\langle X\rangle$, it is generated as a field by $k\langle X\rangle$, and it follows that any element $a \in k(X)$ is contained in some $k(Y)$, Y a finite subset of X.

Now, let Y_1, Y_2 be two nonempty disjoint subsets of X. We have that $k(Y_1) \cup k(Y_2) \subseteq k(Y_1 \cup Y_2)$ and since $k(Y_1 \cup Y_2)$ is generated as a field by $k\langle Y_1 \cup Y_2\rangle$, and clearly also by $k\langle Y_1\rangle \cup k\langle Y_2\rangle$, it follows that the subfield of $k(X)$ generated by $k(Y_1) \cup k(Y_2)$ is $k(Y_1 \cup Y_2)$.

Let D be the universal field of fractions of the k-free product $k(Y_1) * k(Y_2)$ (this k-free product is a fir). The ring $k(Y_1) * k(Y_2)$ contains $k\langle Y_1\rangle * k\langle Y_2\rangle$ as a subring, and we may identify this subring with $k\langle Y_1 \cup Y_2\rangle$. Let E be the subring of $k(Y_1 \cup Y_2)$ generated by $k(Y_1) \cup k(Y_2)$; then there is a canonical homomorphism of $k(Y_1) * k(Y_2)$ onto E. If A is a full matrix over $k\langle Y_1 \cup Y_2\rangle$, then it is invertible in $k(Y_1 \cup Y_2)$, hence it is full over E. By the existence of the homomorphism $k(Y_1) * k(Y_2) \rightarrow E$, it follows that A is full over $k(Y_1) * k(Y_2)$. Hence A is invertible over D. This shows that D contains an honest closure of $k\langle Y_1 \cup Y_2\rangle$ which we shall denote by H. A similar argument shows that H contains honest closures of $k\langle Y_1\rangle$ and $k\langle Y_2\rangle$, hence it contains $k(Y_1)$ and $k(Y_2)$. It follows that $H \supseteq k(Y_1) * k(Y_2)$, and since the field generated by $k(Y_1) * k(Y_2)$ is D, it follows that $H = D$. This implies that the homomorphism $k(Y_1) * k(Y_2) \rightarrow E$ is an isomorphism and E may be identified with $k(Y_1) * k(Y_2)$.

Now, if $a \in k(X)$ and $a \notin k$, then $a \in k(Y)$ for some finite subset Y of X. Since X is infinite, there exists $x \in X - Y$. Taking $Y_1 = Y$ and $Y_2 = \langle x\rangle$ we get that the subring of $k(X)$ generated by $k(Y) \cup k(x)$ may be identified with the k-free product $k(Y) * k(x)$. Hence $ax \neq xa$ and this completes the proof that the center of $k(X)$ is k.

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