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COMPARISON THEOREMS FOR NONSELFADJOINT DIFFERENTIAL EQUATIONS BASED ON INTEGRAL INEQUALITIES

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ABSTRACT. In a variant of the classical Sturmian comparison theorem for selfadjoint Sturm-Liouville equations, A. Ju. Levin has replaced the pointwise conditions on the coefficients by an integral inequality. This theorem is generalized to apply to nonselfadjoint differential equations of the form

$$u'' + b(x)u' + c(x)u = 0.$$

The basic Sturmian comparison theorem deals with functions $u(x)$ and $v(x)$ satisfying

$$u'' + c(x)u = 0, \tag{1}$$

$$v'' + \gamma(x)v = 0. \tag{2}$$

If $\gamma(x) \geqq c(x)$, then solutions of (2) oscillate more rapidly than solutions of (1). More precisely, if $u(x)$ is a nontrivial solution of (1) for which $u(x_1)=u(x_2)=0$ $(x_1<x_2)$ and $\gamma(x)\geqq c(x)$ for $x_1 \leqq x \leqq x_2$, then $v(x)$ has a zero in $(x_1, x_2]$.

This basic result has numerous generalizations, the following of which will be relevant to this paper.

1. It is possible to replace the condition $u(x_1)=0$ by $u'(x_1)+\sigma u(x_1)=0$ where σ is a constant $(-\infty \leqq \sigma < \infty)$ and $\sigma=-\infty$ is used to denote the condition $u(x_1)=0$. In this case one concludes that every solution of (2) satisfying $v'(x_1)+\tau v(x_1)=0$ with $\tau \geqq \sigma$ has a zero in $(x_1, x_2]$.

2. The pointwise inequality $\gamma(x) \geqq c(x)$ can be replaced by weaker integral inequalities. Such weaker conditions have been established by several authors, but of special interest to us here are the results of Levin [1] which deal with nontrivial solutions of (1) and (2) satisfying

$$u'(x_1) + \sigma u(x_1) = 0, \tag{3}$$

$$v'(x) + \tau v(x_1) = 0, \tag{4}$$

respectively.

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THEOREM 1 (LEVIN). *If σ and τ are finite, $u(x_2)=0$ for some $x_2>x_1$, and if*

$$\tau + \int_{x_1}^{x} \gamma(t)\,dt \geqq \left|\sigma + \int_{x_1}^{x} c(t)\,dt\right|$$

for $x_1 \leqq x \leqq x_2$, then $v(x)$ has a zero in $(x_1, x_2]$.

3. Sturmian theorems have been studied for the general *nonselfadjoint* linear differential equation of second order. Such results have been established by Kamke [2] using the Pruefer transformation and by Swanson [3] and the author [4] as special cases of Sturmian theorems for nonselfadjoint elliptic equations. However the hypotheses required in these theorems were either pointwise inequalities or integral inequalities of a type different from Levin's and involving the solution $u(x)$ of the differential equation which "oscillates slower."

The purpose of this paper is to unify the three generalizations indicated above for the case of nonselfadjoint equations of the form

$$u'' - 2b(x)u' + c(x)u = 0, \tag{5}$$

$$v'' - 2\beta(x)v' + \gamma(x)v = 0 \tag{6}$$

whose coefficients are assumed to be real and continuous. (The general linear second order differential equation can always be put into such form by use of a Liouville transformation.) By means of the transformation

$$w = -u'/u; \qquad z = -v'/v;$$

the equations (5) and (6) are transformed into Riccati equations

$$w' = w^2 + 2bw + c, \tag{5'}$$

$$z' = z^2 + 2\beta z + \gamma \tag{6'}$$

and the initial conditions

$$-u'(x_1)/u(x_1) = \sigma; \qquad -v'(x_1)/v(x_1) = \tau \tag{7}$$

for (5) and (6) become initial values

$$w(x_1) = \sigma; \qquad z(x_1) = \tau \tag{8}$$

for (5′) and (6′). The differential equations (5′) and (6′) subject to (8) can in turn be written as integral equations

$$w(x) = \sigma + \int_{x_1}^{x} w^2\,dt + \int_{x_1}^{x} 2bw\,dt + \int_{x_1}^{x} c\,dt, \tag{5''}$$

$$z(x) = \tau + \int_{x_1}^{x} z^2\,dt + \int_{x}^{x} 2\beta z\,dt + \int_{x_1}^{x} \gamma\,dt. \tag{6''}$$

It is obvious from these equations that if $\tau \geqq \sigma \geqq 0$, $\beta(x) \geqq b(x) \geqq 0$, and

$$\int_{x_1}^{x} \gamma(t)\, dt \geqq \int_{x_1}^{x} c(t)\, dt \geqq 0$$

on an interval $[x_1, x_2]$, then $z(x) \geqq w(x) \geqq 0$ as long as $z(x)$ can be continued on $[x_1, x_2]$. Since the singularities of $w(x)$ and $z(x)$ correspond to the zeros of $u(x)$ and $v(x)$, respectively, these observations lead to the following elementary comparison theorem for (5) and (6).

THEOREM 2. *Suppose* $u(x)$ *is a nontrivial solution of* (5) *satisfying* $-u'(x_1)/u(x_1) = \sigma \geqq 0$, $u(x_2) = 0$. *If*

(i) $$\beta(x) \geqq b(x) \geqq 0 \qquad \textit{for } x_1 \leqq x \leqq x_2,$$

(ii) $$\int_{x_1}^{x} \gamma(t)\, dt \geqq \int_{x_1}^{x} c(t)\, dt \geqq 0 \quad \textit{for } x_1 \leqq x \leqq x_2,$$

then every solution of (6) *satisfying* $-v'(x_1)/v(x_1) \geqq \sigma$ *has a zero in* $(x_1, x_2]$.

We shall be interested in variations of Theorem 2 which do not require the nonnegativity of σ, τ, $b(x)$, and $\int_{x_1}^{x} c(t)\, dt$. To that end we note that the integral equations (5″) and (6″) can be written

(5″) $$w(x) = \sigma + \int_{x_1}^{x} (w + b)^2\, dt + \int_{x_1}^{x} (c - b^2)\, dt,$$

(6″) $$z(x) = \tau + \int_{x_1}^{x} (z + \beta)^2\, dt + \int_{x_1}^{x} (\gamma - \beta^2)\, dt.$$

This formulation shows that condition (ii) of Theorem 2 can be replaced by

$$\int_{x_1}^{x} (\gamma - \beta^2)\, dt \geqq \int_{x_1}^{x_2} (c - b^2)\, dt \geqq 0.$$

It also allows for other results of a more general nature.

LEMMA 1. *Let* $w(x)$ *and* $z(x)$ *be solutions of* (5″) *and* (6″), *respectively, for which* $\sigma > -\infty$ *and*

(i) $$\tau + \int_{x_1}^{x} (\gamma - \beta^2)\, dt > \left| \sigma + \int_{x_1}^{x} (c - b^2)\, dt \right| \quad \textit{for } x_1 \leqq x \leqq x_2,$$

(ii) $$\beta(x) \geqq |b(x)| \qquad \textit{for } x_1 \leqq x \leqq x_2.$$

Then $z(x) > |w(x)|$ *as long as* $z(x)$ *can be continued on* $[x_1, x_2]$.

PROOF. From (6″) we have $z(x) \geqq \tau + \int_{x_1}^{x} (\gamma - \beta^2)\, dt$ for $x_1 \leqq x \leqq x_2$. Using

(i) and (5″) this implies that

$$z(x) > -\sigma - \int_{x_1}^{x} (c - b^2)\,dt$$
$$> -\sigma - \int_{x_1}^{x} (c - b^2)\,dt - \int_{x_1}^{x} (w + b)^2\,dt > -w(x)$$

for $x_1 \leqq x \leqq x_2$. It remains to show that $z(x) > w(x)$. To that end we assume to the contrary that there exists $x_0 \in (x_1, x_2]$ such that $z(x_0) \leqq w(x_0)$. Then there exists an $\tilde{x} \in (x_1, x_0]$ such that $z(\tilde{x}) = w(\tilde{x})$ and $z(x) > |w(x)|$ for $x_1 \leqq x < \tilde{x}$. Using (ii) we have that

$$z(x) + \beta(x) > |w(x)| + |b(x)| \geqq |w(x) + b(x)| \quad \text{for } x_1 \leqq x < \tilde{x},$$

and consequently that $\int_{x_1}^{\tilde{x}} (z+\beta)^2\,dt > \int_{x_1}^{\tilde{x}} (w+b)^2\,dt$. Using (6″), (i), and (5″) yields

$$w(\tilde{x}) = \sigma + \int_{x_1}^{\tilde{x}} (c + b^2)\,dt + \int_{x_1}^{\tilde{x}} (w + b)^2\,dt$$
$$< \tau + \int_{x_1}^{\tilde{x}} (\gamma - \beta^2)\,dt + \int_{x_1}^{\tilde{x}} (z + \beta)^2\,dt = z(\tilde{x})$$

which is a contradiction and establishes the lemma.

A continuity argument can now be used to establish the following.

LEMMA 2. *Let $w(x)$ and $z(x)$ be solutions of* (5″) *and* (6″), *respectively, for which $\sigma > -\infty$ and*

(i) $\quad \tau + \int_{x_1}^{x} (\gamma - \beta^2)\,dt \geqq \left|\sigma + \int_{x_1}^{x} (c - b^2)\,dt\right| \quad$ *for* $x_1 \leqq x \leqq x_2$,

(ii) $\quad \beta(x) \geqq |b(x)| \quad$ *for* $x_1 \leqq x \leqq x_2$.

Then $z(x) \geqq w(x)$ as long as $z(x)$ can be continued on $[x_1, x_2]$.

As an immediate consequence of Lemma 2 we have the following generalization of Levin's Theorem 1.

THEOREM 3. *Suppose $u(x)$ and $v(x)$ are nontrivial solutions of* (5) *and* (6), *respectively, and that $u(x) \neq 0$ for $x_1 \leqq x < x_2$, $u(x_2) = 0$. If*

(i) $\quad -\dfrac{v'(x_1)}{v(x_1)} + \int_{x_1}^{x} (\gamma - \beta^2)\,dt \geqq \left|-\dfrac{u'(x_1)}{u(x_1)} + \int_{x_1}^{x} (c - b^2)\,dt\right|$

for $x_1 \leqq x \leqq x_2$,

(ii) $\quad \beta(x) \geqq |b(x)| \quad$ *for* $x_1 \leqq x \leqq x_2$,

then $v(x)$ has a zero in $(x_1, x_2]$.

Theorems 2 and 3 show how to estimate zeros of the nonselfadjoint equations (6) in terms of the coefficients and the initial value of $-v'(x_1)/v(x_1)$. Furthermore Theorem 2 shows that such estimates are sometimes simpler in case $\tau \equiv -v'(x_1)/v(x_1)$ is nonnegative. The following result shows how one can "shift" the initial value of $-v'(x_1)/v(x_1)$ by means of a compensating shift in the coefficient of the differential equation.

THEOREM 4. *Let $v(x)$ be a solution of* (6) *satisfying $-v'(x_1)/v(x_1)=\tau$. The first zero of $v(x)$ is the same as the first zero of $V(x)$, where $V(x)$ is a solution of*

$$\begin{gathered} V'' - 2(\beta - \tau_0)V' + (c - 2\beta\tau_0 + \tau_0^2)V = 0, \\ -V'(x_1)/V(x_1) = \tau + \tau_0. \end{gathered} \tag{9}$$

PROOF. The substitution $z(x) = -v'(x)/v(x)$ leads to

$$z(x) = \tau + \int_{x_1}^{x} z^2\,dt + \int_{x_1}^{x} 2\beta z\,dt + \int_{x_1}^{x} c\,dt.$$

Defining $Z(x) = z(x) + \tau_0$ yields

$$\begin{aligned} Z(x) &= \tau + \tau_0 + \int_{x_1}^{x} (Z - \tau_0)^2\,dt + \int_{x_1}^{x} 2\beta(Z - \tau_0)\,dt + \int_{x_1}^{x} c\,dt \\ &= \tau + \tau_0 + \int_{x_1}^{x} Z^2\,dt + \int_{x_1}^{x} 2(\beta - \tau_0)Z\,dt + \int_{x_1}^{x} (c - 2\beta\tau_0 + \tau_0^2)\,dt. \end{aligned}$$

Now the first singularity of $Z(x)$ coincides with the first singularity of $z(x)$ and therefore with the first zero of $v(x)$. But the first singularity of $Z(x)$ also coincides with the first zero of $V(x)$ satisfying (9), by the substitution $Z(x) = -V'(x)/V(x)$. This completes the proof.

The shift formula of Theorem 4 can also be applied in connection with other known comparison theorems for nonselfadjoint differential equations [**4**].

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