PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, July 1972

ON OPERATORS ON SEPARABLE BANACH SPACES WITH ARBITRARY PRESCRIBED POINT SPECTRUM

GERHARD K. KALISCH[1]

ABSTRACT. For every compact subset C of $\boldsymbol{R}$ and every p in $(1, \infty)$ there exists a bounded linear operator acting in a suitable closed subspace of $L_p(0, 1)$ whose spectrum and point spectrum coincide with each other and with C.

The purpose of this note is to exhibit certain separable Banach spaces (including Hilbert space) that admit bounded linear operators with the property that their spectra σ are pure point spectra with σ any prescribed compact subset of the reals $\boldsymbol{R}$.

Consider $L_p(0, 1)=L_p$ with $1<p<\infty$. Let $(Mf)(x)=xf(x)$ and $(Jf)(x)=\int_0^x f(y)\,dy$ for $f\in L_p$.

THEOREM. *Let C be a compact subset of $\boldsymbol{R}$. Then for every p satisfying $1<p<\infty$ there exists a bounded linear operator S acting in a closed subspace $\mathscr{K}$ of $L_p(0, 1)$ whose spectrum coincides with its point spectrum and equals C. The operator S may be chosen to be of the form $\alpha T_1+\beta$ with $\alpha>0$ and β in $\boldsymbol{R}$ where T_1 is the restriction of $T=M-J$ on L_p to a suitable closed invariant subspace $\mathscr{K}$ of T.*

PROOF. It suffices to show that if $C\subset[0, 1)$, then there is an operator T_1 as described in the theorem with spectrum T_1=point spectrum $T_1=C$. Let $\mathscr{K}$ be the closed subspace of L_p spanned by $\Phi_C=\{\varphi_\lambda;\ \lambda\in C\}$ where $T\varphi_\lambda=\lambda\varphi_\lambda$ and φ_λ is the characteristic function of $[\lambda, 1]$. Clearly $\mathscr{K}$ is invariant under T; call T_1 its restriction to $\mathscr{K}$. The inclusion $C\subset$ point spectrum T_1 being immediate, we shall show that spectrum $T_1\subset C$ or rather complement $C\subset$ resolvent set of T_1, which proves the theorem.

Consider first $\zeta_0\notin[0, 1]$. The formula $(M-J-\zeta)^{-1}=M_{t-\zeta}^{-1}+JM_{t-\zeta}^{-2}$ where $M_g f=gf$ shows that $\zeta_0\in$ resolvent set of T, so $(T-\zeta_0)^{-1}$ is a bounded operator on L_p. Let $\mathscr{K}_0$ be the set of all finite linear combinations of the $\varphi_\lambda\in\Phi_C$. We have $(T-\zeta_0)^{-1}\mathscr{K}_0\subset\mathscr{K}_0$ since

$$(T-\zeta_0)^{-1}\sum\gamma_j\varphi_j=\sum(\lambda_j-\zeta_0)^{-1}\gamma_j\varphi_j$$

Received by the editors December 4, 1970 and, in revised form, July 20, 1971.

AMS 1970 *subject classifications*. Primary 47A10; Secondary 46C10, 46E30.

Key words and phrases. Bounded linear operator, L_p spaces, point spectrum.

[1] I wish to acknowledge gratefully National Science Foundation Grant GP-21334.

where $\varphi_j=\varphi_{\lambda_j}$. Consider now $k=\lim k_n\in\mathscr{K}$ with $k_n\in\mathscr{K}_0$. We have $(T-\zeta_0)^{-1}k=(T-\zeta_0)^{-1}\lim k_n=\lim(T-\zeta_0)^{-1}k_n\in\mathscr{K}$ so that $(T-\zeta_0)^{-1}\mathscr{K}\subset\mathscr{K}$. It is now an easy matter to check that the restriction of $(T-\zeta_0)^{-1}$ to $\mathscr{K}$ is an inverse of $T_1-\zeta_0$ so that $\zeta_0\in$ resolvent set of T_1.

If $\zeta_0\notin C$ but $\zeta_0\in[0, 1]$ we can still show that $(T_1-\zeta_0)^{-1}$ exists as a bounded linear operator of $\mathscr{K}$ into itself so that in this case too we have $\zeta_0\in$ resolvent set of T_1. We first observe that if I is an interval in the complement of C in $[0, 1]$, then the functions of $\mathscr{K}$, being limits of linear combinations of characteristic functions of $[\lambda, 1]$ with $\lambda\in C$, are *constant* on I. Consider then $\zeta_0\in I=(\lambda_1, \lambda_2)\subset[0, 1]\backslash C$ and calculate $(T-\zeta_0)^{-1}$ on $\mathscr{K}$, as follows: every function $f_0\in\mathscr{K}$ can be written as $f_0=f_1+f_2$ with $f_1=f_0$ on the complement of I, $f_1=0$ on I; $f_2=\gamma\chi_I$ where γ is the constant value of f_0 on I. We have

$$(T-\zeta_0)^{-1}f_0=(T-\zeta_0)^{-1}f_1+(T-\zeta_0)^{-1}f_2=g_1+g_2$$

with

$$g_1(t)=(t-\zeta_0)^{-1}f_1+\int_0^t\frac{f_1(s)}{(s-\zeta_0)^2}\,ds$$

and

$$g_2(t)=0 \quad \text{on } [0, \lambda_1],$$

$$g_2(t)=\frac{-\gamma}{\zeta_0-\lambda_1} \quad \text{on } I=(\lambda_1, \lambda_2),$$

$$g_2(t)=\frac{-\gamma(\lambda_2-\lambda_1)}{(\lambda_2-\zeta_0)(\zeta_0-\lambda_1)} \quad \text{on } [\lambda_2, 1].$$

This may be seen as follows:

(1) Observe that on $\mathscr{K}$, the operator $T-\zeta_0$ is 1-1; otherwise there would exist a nonzero function $f\in\mathscr{K}$ such that $Tf=\zeta_0 f$ but the only functions with this property are multiples of φ_{ζ_0} which are *not* in $\mathscr{K}$ as the functions in $\mathscr{K}$ are constant on each interval in $[0, 1]\backslash C$.

(2) We verify that $(T-\zeta_0)(g_0+g_1)=f_1+f_2=f_0$ by means of a simple calculation. We have $\|g_j\|\leqq M_j\ \|f\|$ where M_j does not depend on f_0 but only on ζ_0 and the λ's. Thus $(T-\zeta_0)^{-1}$ is bounded on $\mathscr{K}$. Since

$$(T-\zeta_0)^{-1}\varphi_\lambda=\frac{1}{\lambda-\zeta_0}\,\varphi_\lambda \quad \text{for } \lambda\neq\zeta_0,$$

we have $(T-\zeta_0)^{-1}\mathscr{K}_0\subset\mathscr{K}_0$ and we conclude as before that $\zeta_0\notin$ spectrum T_1. This concludes the proof of the theorem.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92664