PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, July 1972

MAXIMAL OPERATORS AND CAPACITY

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ABSTRACT. It is shown that many maximal functions defined on the L_p spaces are bounded operators on L_p if and only if they satisfy a capacitary weak type inequality.

1. Throughout X will denote either the n-dimensional torus or Euclidean $\boldsymbol{R}^n$. For each $\varepsilon>0$ and $r\in I$, I some index set, let $\theta_r^\varepsilon(x)$ be a complex valued function of $x\in X$ belonging to $L_q(X)$, for all q, $1<q<\infty$. Furthermore, assume that for each fixed $r\in I$, $\theta_r^\varepsilon * f\to F_r$ strongly in $L_p(X)$ as $\varepsilon\to 0$, $f\in L_p(X)$, $1<p<\infty$. The symbol $*$ denotes the usual convolution integration over X. Thinking of θ_r^ε as an approximation to a singular integral (convolution) operator θ_r on $L_p(X)$ for each $r\in I$, we write $F_r=\theta_r * f$. If $\theta_r\in L_q$, for all q, $1<q<\infty$, we permit I to be arbitrary, otherwise assume I is countable. Define the maximal operator $M(f)(x)$ by:

$$M(f)(x)=\sup_{r\in I}|\theta_r * f(x)| \tag{1}$$

and assume $M(f)$ is Lebesgue measurable on X. The theory for θ_r^ε, θ_r, and M given here is modeled after the various maximal operators occurring in the literature, in particular when $\theta_r\in L_q$, for all q, $1<q<\infty$, θ_r could be an "average" (as with the Hardy-Littlewood maximal function on $\boldsymbol{R}^1$) or a Dirichlet kernel (as with regard to "partial sums" of multiple Fourier series—see [4], [6], [9]). When θ_r is singular, interesting examples are

$$\theta_n(x)=x^{-1}e^{-inx},\qquad x\in X,\ n=\text{integer},$$

or the various "modified Dirichlet" kernels appearing in [9]. In general such operators satisfy a maximal inequality:

$$\|M(f)\|_{p,X}\leqq C_p\,\|f\|_{p,X} \tag{2}$$

Received by the editors July 18, 1971.

AMS 1970 *subject classifications*. Primary 31B15, 42A20, 42A92; Secondary 43A15, 44A25.

Key words and phrases. Capacity, maximal functions, Fourier series, singular integrals, L_p spaces.

[1] This research was partially supported by NSF GP 23400.

for all $f \in L_p(X)$, $1<p<\infty$, C_p independent of f. The norms are $L_p(X)$ norms.

The main result states that inequalities of type (2) are equivalent to a *capacitary weak type* inequality. This is Theorem 1 of §2. In §3 some remarks concerning the results appearing in [3] and [8] are made, in particular it is possible to view these as corresponding to our case $p=2$.

2. By a capacity C we will mean a nonnegative extended real valued set function defined on a σ-additive class of sets of X which contains the compact sets and such that (i) $C(\varnothing)=0$, $\varnothing$ =empty set, (ii) $C(A)\leqq C(B)$, for $A\subset B$, (iii) $C(\bigcup_i A_i)\leqq \sum_i C(A_i)$. In particular, we will be interested in:

DEFINITION. For $k \in L_1^+(\boldsymbol{R}^n)$ and lower semicontinuous, let

$$C_{k,p}(A) = \inf \|f\|_{p,X}^p, \qquad 1 < p < \infty,$$

where the infimum is over all $f\in L_p^+(X)$ such that $k*f(x)\geqq 1$ on A, $A\subset X$.

The set functions $C_{k,p}$ are capacities and have been studied extensively in [7]. A property of $C_{k,p}$ proved in [7] and used below is: if $f_k\to f$ strongly in L_p, $k=1, 2, \cdots$, then there is a subsequence f_k' such that $k*f_k'\to k*f$, $C_{k,p}$—a.e.

THEOREM 1. *Suppose M is the maximal operator defined in* §1 *and satisfies* (2), $1<p<\infty$. *Then*

$$C_{k,p}[M(k * f) > t] \leqq C_p^p t^{-p} \, \|f\|_{p,X}^p, \qquad t > 0, \tag{3}$$

for all $f\in L_p(X)$, $1<p<\infty$. *Conversely, if* (3) *holds, then* (2) *holds for some constant* C_p' *independent of* f, $1<p<\infty$.

We say that $M(k*\cdot)$ is of capacitary weak type (p,p) when (3) holds. An important feature in (3) is that C_p is independent of both f *and* k. This allows us to view the usual weak type inequality

$$m[M(f) > t] \leqq C_p^p t^{-p} \, \|f\|_{p,X}^p,$$

m=Lebesgue measure, as a limiting case of (3) in light of

THEOREM 2. *Let* $k_i\geqq 0$, $i=1, 2, \cdots$, *and satisfy*

(a) $\int_{\boldsymbol{R}^n} k_i(t)\,dt\to 1$, *as* $i\to\infty$,

(b) $\int_{|t|\geqq\eta} k_i(t)\,dt\to 0$, *as* $i\to\infty$, *for all* $\eta>0$,

then $C_{k_i,p}(K)\to m(K)$, *K compact in X.*

PROOF OF THEOREM 1. If $\theta_r \in L_q$, $1<q<\infty$, then

$$\theta_r * (k * f)(x) = k * (\theta_r * f)(x)$$

for all $x\in X$. Otherwise, if $\theta_r^\varepsilon * f\to\theta_r * f$ in L_p, then there is a sequence

$\varepsilon_j \to 0$ as $j \to \infty$ such that

$$\theta_r^{\varepsilon_j} * (k * f)(x) = k * (\theta_r^{\varepsilon_j} * f)(x) \to k * (\theta_r * f)(x),$$

$C_{k,p}$—a.e. Hence $\theta_r*(k*f)(x)=k*(\theta_r*f)(x)$, $C_{k,p}$—a.e. on X. Of course the exceptional set depends on r. Call it E_r. Then

$$|\theta_r * (k * f)(x)| \leqq k * |\theta_r * f|\ (x), \qquad x \notin E_r,$$

and $C_{k,p}(E_r)=0$. Hence

$$M(k * f)(x) \leqq k * M(f)(x), \qquad C_{k,p}\text{—a.e.}$$

since $C_{k,p}(\bigcup_{r\in I} E_r) \leqq \sum_{r \in I} C_{k,p}(E_r)=0$. It now follows easily from the definition of $C_{k,p}$ that

$$C_{k,p}[M(k*f) > t] \leqq C_{k,p}[k * M(f) > t] \leqq t^{-p} \|M(f)\|_{p,X}^p \leqq C_p^p t^{-p} \|f\|_{p,X}^p.$$

For the converse, first note that if $k*f \geqq 1$ on A,

$$m(A) \leqq \int_A k * f\, dx \leqq \|k * f\|_{p,X} \cdot m(A)^{1/p'} \leqq \|k\|_1 \|f\|_{p,X}\, m(A)^{1/p'}.$$

Thus if $m(A) < \infty$, $m(A) \leqq \|k\|_1^p C_{k,p}(A)$. This inequality also holds when $m(A)=\infty$ since $C_{k,p}(A)=\infty$ as a result. We now choose $k=g_\alpha$, where g_α, $\alpha>0$, is the L_1^+ function on $\boldsymbol{R}^n$ which is the Fourier transform of $(2\pi)^{-n/2}(1+|\xi|^2)^{-\alpha/2}$, $\xi \in \boldsymbol{R}^n$. Clearly $\|g_\alpha\|_1=1$. Thus

$$m[M(g_\alpha * f) > t] \leqq C_{g_\alpha,p}[M(g_\alpha * f) > t] \leqq C_p^p t^{-p} \|f\|_{p,X}^p.$$

Note, $g_\alpha*f(x)-g_\beta*f(x)=g_\sigma*[g_{\alpha-\sigma}*f-g_{\beta-\sigma}*f](x)$, $g_0*f=f$, $\alpha, \beta>0$, $\sigma=\min(\alpha, \beta)$. Thus

$$m[M(g_\alpha * f - g_\beta * f) > t] \leqq C_p^p t^{-p} \|g_{\alpha-\sigma} * f - g_{\beta-\sigma} * f\|_p^p.$$

Since $g_\delta*f \to f$ strongly in L_p as $\delta \to 0^+$ (g_δ is an approximation of the identity—see [2] for the necessary properties of g_δ), we have, by the sublinearity of M, $M(g_\alpha*f)$ is Cauchy in measure as $\alpha \to 0$. Thus $M(g_\alpha*f) \to h$ in measure as $\alpha \to 0$ and $m[h>t] \leqq C_p^p t^{-p} \|f\|_{p,X}^p$. But $\theta_r*(g_\alpha*f)(x) \leqq M(g_\alpha*f)(x)$, all x, and $\theta_r*(g_\alpha*f)(x) \to \theta_r*f(x)$, all x, when $\theta_r \in L_q$, $1<q<\infty$, *or* only a.e. for some sequence $\alpha_j \to 0$, when θ_r is singular. Hence

$$\theta_r * f(x) \leqq h(x), \qquad x \notin E_r,\ m(E_r) = 0.$$

But then $M(f)(x) \leqq h(x)$, a.e. Thus $M(\cdot)$ is of usual weak type (p,p), $1<p<\infty$, and the desired result follows by the Marcinkiewicz interpolation theorem. Q.E.D.

COROLLARY. *$\int M(k*f)\,d\mu \leqq C_p \|f\|_{p,X} \cdot I_{k,p}(\mu)^{1/p'}$ is equivalent to* (2), *$1<p<\infty$, where $I_{k,p}(\mu)$ is the generalized energy of μ, μ a positive Borel measure, i.e.,*

$$I_{k,p}(\mu) = \int k * (k * \mu)^{1/(p-1)}\, d\mu.$$

See [7].

PROOF OF THEOREM 2. We have already noted $m(K) \leqq \|k\|_1^p C_{k,p}(K)$, hence $m(K) \leqq \liminf_{i\to\infty} C_{k_i,p}(K)$.

For the other inequality, note that for the given K we can take a bounded open set $G \supset K$ and from (b): for each ε, $0<\varepsilon<1$, there is an i_0 such that $k_i * \chi_G(x) \geqq 1-\varepsilon$ on K, when $i \geqq i_0$, χ_G=characteristic function of G. Thus

$$C_{k_i,p}(K) \leqq m(G)/(1-\varepsilon)^p,$$

$$\therefore \quad \limsup_{i\to\infty} C_{k_i,p}(K) \leqq m(G)/(1-\varepsilon)^p,$$

and by the arbitrariness of G and ε the result follows.

REMARK. If $\theta_r * f \to f$, a.e., for f in a dense class of L_p and (3) holds, then $\theta_r*(k*f) \to k*f$, $C_{k,p}$—a.e. for all $f \in L_p$, $1<p<\infty$.

3. For $k \in L_1^+(\boldsymbol{R}^n)$, define

$$c_{k,p}(K) = \sup \|\mu\|_1, \qquad 1 \leqq p < \infty,$$

where the supremum is over all nonnegative Radon measures μ ($\mu \in \mathscr{M}^+$) for which supp $\mu \subset K$, K compact, and $\|k*\mu\|_{p',X} \leqq 1$. Here $\|\mu\|_1$ is the total variation of μ. The capacity $c_{k,p}$ is also studied in [7] where, for all analytic sets A, $c_{k,p}(A) = C_{k,p}(A)^{1/p}$ is established, $1<p<\infty$. For $p=1$, $C_{k,1}$ is defined using $\mathscr{L}_1^+$ in place of L_p^+, i.e., $\mu \in \mathscr{M}^+$ with $\|\mu\|_1 < \infty$. For compact K, $C_{k,1}(K) = c_{k,1}(K)$. See [5]. Thus when $p=2$ and when it is possible to write $k = k_0 * k_0$, $k_0 \in L_1^+$, $k_0(x) = k_0(|x|)$ and decreasing as $|x| \uparrow \infty$, then

$$C_{k,1}(A) \sim C_{k_0,2}(A), \quad \text{all } A \subset X.$$

Here the symbol $\sim$ means the ratio is bounded above and below by positive finite constants independent of A. (Use the "boundedness principle" of [1] to prove this.)

Also, if $\mathfrak{F}$ denotes the Fourier transform in $\boldsymbol{R}^n$, and H_k consists of all $u \in L_2(X)$ such that

$$\|u\|_k = \left(\sum_{-\infty}^{\infty} |\hat{u}(n)|^2 \, |\mathfrak{F}k(n)|^2\right)^{1/2} < \infty,$$

$\hat{}$ denoting Fourier coefficients, then each $u \in H_k$ has an L_2 representative as $k*f$, $f\in L_2(X)$, and $\|u\|_k=\|f\|_{2,X}$. With these facts, it is possible to view the results of [3] and [8] as included in the case $p=2$ of Theorem 1.

Finally, we note that if $k=g_\alpha$, the classical theorem of Beurling-Salem-Zygmund becomes

$$S_N(g_\alpha * f)(x) \to g_\alpha * f(x), \qquad C_{g_\alpha,2}\text{—a.e.}$$

S_N the Nth partial sum of the Fourier series of $g_\alpha * f$, $f\in L_2(-\pi,\pi)$. See [**10**, p. 195].

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