MAXIMAL OPERATORS AND CAPACITY

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ABSTRACT. It is shown that many maximal functions defined on the $L_p$ spaces are bounded operators on $L_1$ if and only if they satisfy a capacitary weak type inequality.

1. Throughout $X$ will denote either the $n$-dimensional torus or Euclidean $\mathbb{R}^n$. For each $\epsilon > 0$ and $r \in I$, $I$ some index set, let $\theta_r^\epsilon(x)$ be a complex valued function of $x \in X$ belonging to $L_q(X)$, for all $q$, $1 < q < \infty$. Furthermore, assume that for each fixed $r \in I$, $\theta_r^\epsilon \ast f \to F_r$ strongly in $L_p(X)$ as $\epsilon \to 0$, $f \in L_p(X)$, $1 < p < \infty$. The symbol $\ast$ denotes the usual convolution integration over $X$. Thinking of $\theta_r^\epsilon$ as an approximation to a singular integral (convolution) operator $\theta_r$ on $L_q(X)$ for each $r \in I$, we write $F_r = \theta_r \ast f$. If $\theta_r \in L_q$, for all $q$, $1 < q < \infty$, we permit $I$ to be arbitrary, otherwise assume $I$ is countable. Define the maximal operator $M(f)(x)$ by:

$$M(f)(x) = \sup_{r \in I} |\theta_r \ast f(x)|$$

and assume $M(f)$ is Lebesgue measurable on $X$. The theory for $\theta_r^\epsilon$, $\theta_r$, and $M$ given here is modeled after the various maximal operators occurring in the literature, in particular when $\theta_r \in L_q$, for all $q$, $1 < q < \infty$, $\theta_r$ could be an “average” (as with the Hardy-Littlewood maximal function on $\mathbb{R}^1$) or a Dirichlet kernel (as with regard to “partial sums” of multiple Fourier series—see [4], [6], [9]). When $\theta_r$ is singular, interesting examples are

$$\theta_r(x) = x^{-1}e^{-inx}, \quad x \in X, n = \text{integer},$$

or the various “modified Dirichlet” kernels appearing in [9]. In general such operators satisfy a maximal inequality:

$$\|M(f)\|_{p,X} \leq C_p \|f\|_{p,X}$$

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for all \( f \in L_p(X) \), \( 1 < p < \infty \), \( C_p \) independent of \( f \). The norms are \( L_p(X) \) norms.

The main result states that inequalities of type (2) are equivalent to a capacitary weak type inequality. This is Theorem 1 of §2. In §3 some remarks concerning the results appearing in [3] and [8] are made, in particular it is possible to view these as corresponding to our case \( p=2 \).

2. By a capacity \( C \) we will mean a nonnegative extended real valued set function defined on a \( \sigma \)-additive class of sets of \( X \) which contains the compact sets and such that (i) \( C(\emptyset) = 0 \), \( \emptyset = \) empty set, (ii) \( C(A) \leq C(B) \), for \( A \subseteq B \), (iii) \( C(\bigcup_i A_i) \leq \sum_i C(A_i) \). In particular, we will be interested in:

**Definition.** For \( k \in L_1^+(R^n) \) and lower semicontinuous, let

\[
C_k, p(A) = \inf \| f \|_{p, X}^p, \quad 1 < p < \infty,
\]

where the infimum is over all \( f \in L_p^+(X) \) such that \( k \ast f(x) \geq 1 \) on \( A, A \subseteq X \).

The set functions \( C_{k, p} \) are capacities and have been studied extensively in [7]. A property of \( C_{k, p} \) proved in [7] and used below is: if \( f_k \rightarrow f \) strongly in \( L_p \), \( k=1, 2, \cdots \), then there is a subsequence \( f'_k \) such that \( k \ast f'_k \rightarrow k \ast f, \ C_{k, p} \) a.e.

**Theorem 1.** Suppose \( M \) is the maximal operator defined in §1 and satisfies (2), \( 1 < p < \infty \). Then

\[
C_{k, p}[M(k \ast f) > t] \leq C_p^p t^{-p} \| f \|_{p, X}^p, \quad t > 0,
\]

for all \( f \in L_p(X) \), \( 1 < p < \infty \). Conversely, if (3) holds, then (2) holds for some constant \( C_p \) independent of \( f \), \( 1 < p < \infty \).

We say that \( M(k \ast \cdot) \) is of capacitary weak type \((p, p)\) when (3) holds. An important feature in (3) is that \( C_p \) is independent of both \( f \) and \( k \). This allows us to view the usual weak type inequality

\[
m[M(f) > t] \leq C_p^p t^{-p} \| f \|_{p, X}^p,
\]

\( m = \) Lebesgue measure, as a limiting case of (3) in light of

**Theorem 2.** Let \( k_i \geq 0, i=1, 2, \cdots \), and satisfy

(a) \( \int_{R^n} k_i(t) \, dt \rightarrow 1, \) as \( i \rightarrow \infty \),

(b) \( \int_{|t| \geq \eta} k_i(t) \, dt \rightarrow 0, \) as \( i \rightarrow \infty \), for all \( \eta > 0 \),

then \( C_{k_i, p}(K) \rightarrow m(K), K \) compact in \( X \).

**Proof of Theorem 1.** If \( \theta_r \in L_q, 1 < q < \infty \), then

\[
\theta_r \ast (k \ast f)(x) = k \ast (\theta_r \ast f)(x)
\]

for all \( x \in X \). Otherwise, if \( \theta_r \ast f \rightarrow \theta_r \ast f \) in \( L_p \), then there is a sequence
\( \varepsilon, \rightarrow 0 \) as \( j \rightarrow \infty \) such that
\[
\theta_{\varepsilon}^{(k \ast f)}(x) = k \ast (\theta_{\varepsilon}^{(k \ast f)})(x) \rightarrow k \ast (\theta_{\varepsilon} \ast f)(x),
\]
\( C_{k,p} \) a.e. Hence \( \theta_{\varepsilon} \ast (k \ast f)(x) = k \ast (\theta_{\varepsilon} \ast f)(x) \), \( C_{k,p} \) a.e. on \( X \). Of course the exceptional set depends on \( r \). Call it \( E_{r} \). Then
\[
|\theta_{r} \ast (k \ast f)(x)| \leq k \ast |\theta_{r} \ast f| (x), \quad x \notin E_{r},
\]
and \( C_{k,p}(E_{r}) = 0 \). Hence
\[
M(k \ast f)(x) \leq k \ast M(f)(x), \quad C_{k,p} \quad \text{a.e.}
\]
since \( C_{k,p}(\bigcup_{r \in I} E_{r}) \leq \sum_{r \in I} C_{k,p}(E_{r}) = 0 \). It now follows easily from the definition of \( C_{k,p} \) that
\[
C_{k,p}[M(k \ast f) > t] \leq C_{k,p}[k \ast M(f) > t] \leq t^{-p} \|M(f)\|_{p,X} \leq C_{p}^{p} t^{-p} \|f\|_{p,X}^{p}.
\]
For the converse, first note that if \( k \ast f \geq 1 \) on \( A \),
\[
m(A) \leq \int_{A} k \ast f \, dx \leq \|k \ast f\|_{p,X} \cdot m(A)^{1/p'} \leq \|k\|_{1} \|f\|_{p,X} m(A)^{1/p'}.
\]
Thus if \( m(A) > \infty \), \( m(A) \leq \|k\|_{1} C_{k,p}(A) \). This inequality also holds when \( m(A) = \infty \) since \( C_{k,p}(A) = \infty \) as a result. We now choose \( k = g_{\alpha} \), where \( g_{\alpha}, \alpha > 0 \), is the \( L_{1}^{1} \) function on \( R^{n} \) which is the Fourier transform of \((2\pi)^{-n/2}(1 + |\xi|^{2})^{-n/2}, \xi \in R^{n} \). Clearly \( \|g_{\alpha}\|_{1} = 1 \). Thus
\[
m[M(g_{\alpha} \ast f) > t] \leq C_{k,p}[M(g_{\alpha} \ast f) > t] \leq C_{p}^{p} t^{-p} \|f\|_{p,X}^{p}.
\]
Note, \( g_{\alpha} \ast f(x) - g_{\beta} \ast f(x) = g_{\alpha} \ast [g_{\alpha - \beta} \ast f - g_{\beta} \ast f](x), \alpha, \beta > 0, \sigma = \min(\alpha, \beta). \) Thus
\[
m[M(g_{\alpha} \ast f - g_{\beta} \ast f) > t] \leq C_{p}^{p} t^{-p} \|g_{\alpha - \beta} \ast f - g_{\beta} \ast f\|_{p}.
\]
Since \( g_{\alpha} \ast f \) strongly in \( L_{p} \) as \( \delta \rightarrow 0^{+} \) (\( g_{\delta} \) is an approximation of the identity—see [2] for the necessary properties of \( g_{\delta} \)), we have, by the sublinearity of \( M, M(g_{\alpha} \ast f) \) is Cauchy in measure as \( \alpha \rightarrow 0 \). Thus \( M(g_{\alpha} \ast f) \rightarrow h \) in measure as \( \alpha \rightarrow 0 \) and \( m[h > t] \leq C_{p}^{p} t^{-p} \|f\|_{p,X}^{p} \). But \( \theta_{\varepsilon} \ast (g_{\alpha} \ast f)(x) \leq M(g_{\alpha} \ast f)(x), \) all \( x \), and \( \theta_{r} \ast (g_{\alpha} \ast f)(x) \rightarrow \theta_{r} \ast f(x), \) all \( x \), when \( \theta_{\varepsilon} \in L_{q}, 1 < q < \infty \), or only a.e. for some sequence \( \alpha_{j} \rightarrow 0 \), when \( \theta_{\varepsilon} \) is singular. Hence
\[
\theta_{r} \ast f(x) \leq h(x), \quad x \notin E_{r}, m(E_{r}) = 0.
\]
But then \( M(f)(x) \leq h(x) \), a.e. Thus \( M(\cdot) \) is of usual weak type \((p,p), 1 < p < \infty \), and the desired result follows by the Marcinkiewicz interpolation theorem. Q.E.D.
**Corollary.** \( \int M(k*f) \, d\mu \leq C_{k,p} \| f \|_{p', \infty} \cdot I_{k,p}(\mu)^{1/p'} \) is equivalent to (2), \( 1 < p < \infty \), where \( I_{k,p}(\mu) \) is the generalized energy of \( \mu \), \( \mu \) a positive Borel measure, i.e.,

\[
I_{k,p}(\mu) = \int k \ast (k \ast \mu)^{1/(p-1)} \, d\mu.
\]

See [7].

**Proof of Theorem 2.** We have already noted \( m(K) \leq \| k \|_{p}^p C_{k,p}(K) \), hence \( m(K) \leq \lim inf_{i \to \infty} C_{k,i}(K) \).

For the other inequality, note that for the given \( K \) we can take a bounded open set \( G \supseteq K \) and from (b): for each \( \varepsilon, 0 < \varepsilon < 1 \), there is an \( i_0 \) such that \( k \ast \chi_G(x) \geq 1 - \varepsilon \) on \( K \), when \( i \geq i_0 \), \( \chi_G = \text{characteristic function of } G \). Thus

\[
C_{k,i}(K) \leq m(G)/(1 - \varepsilon)^p,
\]

\[
\therefore \quad \lim sup_{i \to \infty} C_{k,i}(K) \leq m(G)/(1 - \varepsilon)^p,
\]

and by the arbitrariness of \( G \) and \( \varepsilon \) the result follows.

**Remark.** If \( \theta_* f \to f, \text{ a.e., for } f \text{ in a dense class of } L_p \) and (3) holds, then \( \theta_*(k*f) \to k*f, C_{k,p} - \text{a.e. for all } f \in L_p, 1 < p < \infty \).

3. For \( k \in L_1^+(\mathbb{R}^n) \), define

\[
c_{k,p}(K) = \sup \| \mu \|_1, \quad 1 \leq p < \infty,
\]

where the supremum is over all nonnegative Radon measures \( \mu \) \( \mu \in \mathcal{M}^+ \) for which \( \text{supp } \mu \subseteq K \), \( K \) compact, and \( \| k \ast \mu \|_{p', \infty} \leq 1 \). Here \( \| \mu \|_1 \) is the total variation of \( \mu \). The capacity \( c_{k,p} \) is also studied in [7] where, for all analytic sets \( A \), \( c_{k,p}(A) = C_{k,p}(A)^{1/p} \) is established, \( 1 < p < \infty \). For \( p = 1 \), \( C_{k,1} \) is defined using \( L_1^+ \) in place of \( L_1^p \), i.e., \( \mu \in \mathcal{M}^+ \) with \( \| \mu \|_1 < \infty \). For compact \( K \), \( C_{k,1}(K) = C_{k,1}(K) \). See [5]. Thus when \( p = 2 \) and when it is possible to write \( k = k_0 \ast k_0, k_0 \in L_1^+, k_0(x) = k_0(|x|) \) and decreasing as \( |x| \uparrow \infty \), then

\[
C_{k,1}(A) \sim C_{k,p,2}(A), \quad \text{all } A \subseteq X.
\]

Here the symbol \( \sim \) means the ratio is bounded above and below by positive finite constants independent of \( A \). (Use the “boundedness principle” of [1] to prove this.)

Also, if \( \mathcal{F} \) denotes the Fourier transform in \( \mathbb{R}^n \), and \( H_k \) consists of all \( u \in L_2(X) \) such that

\[
\| u \|_k = \left( \sum_{n=-\infty}^{\infty} |\hat{u}(n)|^2 / |\mathcal{F}k(n)|^2 \right)^{1/2} < \infty,
\]

\( \hat{u} \) denoting Fourier coefficients, then each \( u \in H_k \) has an \( L_2 \) representative.
as \( k \ast f, f \in L_p(X) \), and \( \| u \|_k = \| f \|_{2, X} \). With these facts, it is possible to view the results of [3] and [8] as included in the case \( p = 2 \) of Theorem 1.

Finally, we note that if \( k = g_x \), the classical theorem of Beurling-Salem-Zygmund becomes

\[
S_N(g_x \ast f)(x) \to g_x \ast f(x), \quad C_{g_x, x} \text{-a.e.}
\]

\( S_N \) the \( N \)th partial sum of the Fourier series of \( g_x \ast f, f \in L_p(-\pi, \pi) \). See [10, p. 195].

**References**


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