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A NOTE ON MODEL COMPLETE MODELS AND GENERIC MODELS[1]

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ABSTRACT. We prove that there are many maximum model complete (= generic) models, and that there exists an (uncountable) theory with no generic models.

After Barwise and Robinson [**1**] we say a model M, of a (first-order) theory T, completes T if every extension of M, which is a model of T, is an elementary extension of M. (By [**1**, Theorem 3.4, p. 129], M completes T^f iff it is T-generic.) It is known

LEMMA 1. *If M completes T and N is an elementary submodel of M, then N also completes T (it follows from Theorem* 1.2).

For a cardinal λ let $Mc(\lambda)$ be the least cardinal κ, such that for all T of power $\leqq\lambda$, *if* T is completed by some model of power κ, *then* for all $\mu\geqq\lambda$ there is a model which completes T and whose power is $\geqq\mu$.

THEOREM 2. $Mc(\lambda)=\mu_\lambda$ (*=the Hanf number of omitting a type*).

REMARK. For the values of μ_λ see, e.g., Chang [**2**, §2, p. 47]; he denotes μ_λ by m_λ.

THEOREM 3. *For arbitrarily large cardinals κ smaller than the first measurable cardinal there exists a complete and countable T and a model M of power κ which complete T, and no proper extension of M completes T.*

Answering Question 8.1 of [**1**] we prove in §2:

THEOREM 4. *There is an uncountable theory T with no T-generic model.*

(This was also proved, independently, by P. Henrard, and later by Macintyre.) Only in §2 knowledge of [**1**] is assumed.

NOTATION. $|M|$ is the universe of the model M. $|A|$ is the cardinality of the set A (so $|L|$ is the number of formulas of L). $\|M\|$ is the cardinality of (the universe of) M. Infinite cardinals are denoted by λ, μ, κ.

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1. THEOREM 1.1.[2] *Let T be a theory, p a type in a language L, and M an infinite model of T which omits p. Then there are a language L_1, a theory T_1 and a type p_1 in L_1 and a model M_1 of T_1 which omits p_1 such that:*

(a) $|L_1| \leqq |L| + \aleph_0$, *$T_1$ is complete.*

(b) *M_1 completes T_1, it omits p_1 and $\|M_1\| = \|M\|$.*

(c) *A model of T_1 completes T_1 iff it omits p_1.*

(d) *If every extension of M which is a model of T realizes p then no extension of M_1 completes T_1.*

(e) *If T has no model of cardinality λ which omits p then there is no model which completes T_1 in cardinality λ.*

PROOF. Without loss of generality assume in L there are no function symbols. Let $p = \{\phi_i(x) : i < |p|\}$. Let us choose infinite disjoint subsets of $|M|$, A_i, $i < |p|$, such that $|M| = \bigcup A_i$.

We expand M to a model M^1 by adding the following relations:

(1) $P_i^{M^1} = A_i$ for every $i < |p|$ (i.e., $P_i^{M^1}$ is a one-place relation, and P_i the corresponding predicate).

(2) A relation R^{M^1} such that $\langle a, b\rangle \in R^{M^1}$ iff there is $i < |p|$, $b \in P_i^{M^1}$, $M \models \neg\phi_i(a)$ and for every $j < i$, $M \models \phi_j[a]$.

(3) An equivalence relation $E_1^{M^1}$ such that: $aE_1^{M^1}b$ iff for some i, $a, b \in P_i^{M^1}$.

Now let us define a model M^2. Its set of elements is

$$|M^2| = \{\langle a, \alpha\rangle : a \in |M^1|, \alpha \leqq \omega\}.$$

Its relations and functions are:

(4) An equivalence relation $E_2^{M^2}$ such that:

$$\langle a, \alpha\rangle E_2^{M^2} \langle b, \beta\rangle \quad \text{iff} \quad a = b.$$

(5) For every n-place relation Q^{M^1} let

$$Q^{M^2} = \{\langle\langle a_1, \alpha_1\rangle, \cdots, \langle a_n, \alpha_n\rangle\rangle : \langle a_1, \cdots, a_n\rangle \in Q^{M^1}; \alpha_1, \cdots, \alpha_n \leqq \omega\}.$$

(6) An equivalence relation $E_3^{M^2}$ such that $\langle a, \alpha\rangle E_3^{M^2} \langle b, \beta\rangle$ iff

(a) $a = b$,

(b) $\alpha = \beta$ or $\alpha = 2n+1$, $\beta = 2n$ or $\alpha = 2n$, $\beta = 2n+1$.

(7) For every $i < |p|$ a function $F_i^{M^2}(x, y)$ such that for every $a, b \in |M^2|$:

(a) if $M^2 \models \neg aE_2b \vee \neg P_i(a)$ then $F_i^{M^2}(a, b) = a$,

(b) if $M^2 \models aE_2b \wedge P_i(a)$ then $M^2 \models P_i(F_i(a, b))$,

(c) if $M^2 \models aE_2b \wedge aE_2c \wedge P_i(a) \wedge b \neq c$ then $M^2 \models \neg E_2(F_i(a, b), F_i(a, c))$.

[2] ADDED IN PROOF (MAY 11, 1972). In Theorem 1.1 if p is countable, we can define M_1 so that $M \equiv N$ implies $M_1 \equiv N_1$. This may help to improve Theorem 3.

Now we define the M_1 we wanted as an expansion of M^2 by:

(8) For every $n<\omega$, $i_1, \cdots, i_n<|p|$ (not necessarily distinct) and formula $\phi(x_1, \cdots, x_n)$ of the language of M^2, we add the relation $R^{M_1}_{\phi, i_1, \cdots, i_n}$ defined by

$$R^{M_1}_{\phi, i_1, \cdots, i_n} = \{\langle a_1, \cdots, a_n\rangle : M^2 \models [P_{i_1}(a_1) \wedge \cdots \wedge P_{i_n}(a_n) \wedge \phi(a_1, \cdots, a_n)]\}.$$

Now let T_1 be the theory of M_1, L_1 its language, and $p_1 = \{\neg P_i(x) : i<|p|\}$. Let us now prove that:

(*) A model of T_1 completes T_1 iff it omits p_1.

By (8) it is clear that every model of T_1 which omits p_1 completes T_1. Suppose now N is a model of T_1 which realizes p_1, and let $a \in |N|$ realize p_1. As N is a model of T_1, by (4) and (6), there are distinct elements c, b_n, $0 \leqq n<\omega$, such that:

$$N \models b_{2n}E_3 b_{2n+1}, \qquad N \models (\forall x)(xE_3c \rightarrow x = c),$$
$$N \models aE_2b_n \quad \text{and} \quad N \models aE_2c \qquad (\text{for every } n).$$

We now define now a submodel N_1 of N, whose set of elements is $|N_1| = |N| - \{c, b_0\}$. Now N_1 is not an elementary submodel of N because

$$N_1 \models (\forall x)(xE_3b_1 \rightarrow x = b_1), \qquad N \models (\exists x)(xE_3b_1 \wedge x \neq b_1).$$

On the other hand N, N_1 are isomorphic: define F by:

$$F(c) = b_1 F(b_n) = b_{n+2} \quad (\text{for } 0 \leqq n < \omega)$$

and

$$F(a^1) = a^1 \quad \text{for } a^1 \in N - \{c_1 b_1, b_2 \cdots\}.$$

Clearly, F is an isomorphism from N onto N_1.

So N, N_1 are models of T, N_1 does not complete T, hence also N does not complete T. So we proved (*).

Now (a) is immediate; (b) follows from the definition of $|M^2| = |M_1|$ and (*); (c) is (*); (d) is clear from (*) and (2); and for (e) we should notice also (7) (which implies that if N is a model of T_1, which omits p_1, then $\|N\|$ is equal to the number of E_2^N-equivalence classes in $|N|$). So we prove the theorem.

The following theorem was already known to Robinson:

THEOREM 1.2. *For every theory T there is a set P of types* (*not all* 1-*types necessarily*) *such that*: *any model M completes T if and only if M is a model of T omitting every type $p \in P$, and* $|P| \leqq |T| + \aleph_0$.

PROOF. Let M be a model, and $|M| = \{a_i | i<\alpha\}$ and Diag M be the set of sentences $\phi(a_i, \cdots, a_n)$ which are satisfied by M where ϕ is a basic formula (=an atomic or negation of an atomic formula). Clearly,

M completes T if and only if $T \cup \text{Diag}\, M$ is a complete theory. By the compactness theorem, this implies: M completes T if and only if: for every formula $\phi(x_1, \cdots, x_m)$ and elements $b_0^1, \cdots, b_m^0 \in |M|$, there are $\phi_1(b_1^1, b_2^1, \cdots), \cdots, \phi_n(b_1^n, b_2^n, \cdots)$ in Diag M such that

$$T \cup \{\phi(b_1^1, \cdots), \cdots, \phi_n(b_1^n, \cdots)\} \vdash \phi(b_1, \cdots, b_n)$$

or, equivalently,

$$T \vdash (\forall \cdots x_j^i \cdots)\Big[\bigwedge_i \phi_i(x_1^i \cdots) \rightarrow \phi(x_1^0, \cdots)\Big]$$

(we should identify the variables x_k^i, x_e^k if $a_i^k = a_e^k$). For every formula $\phi = \phi(x_1, \cdots, x_n)$ let Γ_ϕ be the set of formulas $\theta(x_1, \cdots, x_n, \cdots, x_m)$ which are conjunctions of basic formulas and

$$T \vdash (\forall x_1, \cdots, x_m)[\theta(x_1, \cdots, x_m) \rightarrow \phi(x_1, \cdots, x_n)].$$

Let $p_\phi = \{\neg(\exists x_{n+1}, \cdots, x_m)\theta(x_1, \cdots, x_n, \cdots, x_m) : \theta \in \Gamma_\phi\}$. Clearly, M completes T if and only if for every ϕ, T omits p_ϕ. So $P = \{p_\phi | \phi$ a formula$\}$ satisfies the condition of the theorem.

PROOF OF THEOREM 2. By the definitions of $Mc(\lambda)$, μ_λ, clearly Theorem 1.1 implies $Mc(\lambda) \geqq \mu_\lambda$. Suppose that M completes T, $\|M\| \geqq \mu_\lambda$, $\lambda \geqq |T|$. So by 1.2, M is a model of T and omits every $p \in P$. By, e.g., Chang [**2**, p. 47, (D)], the Hanf number for a sentence in $L_{\lambda^+,\omega}$ is μ_λ, and clearly being a model of T omitting every $p \in P$ can be expressed in $L_{\lambda^+,\omega}$. So T has arbitrarily large models omitting every $p \in P$, hence by 1.2 arbitarily large models completing T. This means $Mc(\lambda) \leqq \mu_\lambda$. So $Mc(\lambda) = \mu_\lambda$.

PROOF OF THEOREM 3. This can be proved using 1.1 and the following (see Malitz and Reinhart [**4**, Theorem XX]).

THEOREM. *For arbitrarily large cardinals λ smaller than the first measurable cardinal, there is a model M_λ, $\|M_\lambda\| = \lambda$, with countable type and with a one place relation P, $P^{M_\lambda} = \{c_n | c_n < \omega\}$, such that: for no proper extension N of M_λ which is elementarily equivalent to M_λ, $P^N = P^{M_\lambda}$.*

(For characterization of those λ which satisfy this, see [**4**].)

2. Let N be the standard model of natural numbers with addition, multiplication and individual constant m for each natural number m. Let $T = Th(N)$, and the language be L^*. Let $K = T \cup \{c_i \neq c_j : i < j < \aleph_1\}$, and its language L, $K_1 = T \cup \{c_i \neq c_j : i < j < \omega\}$ and its language L_1.

THEOREM 2.1. *There is no K-generic model.*

PROOF. It is easy to check that $P(c_1, \cdots, c_m, a_1, \cdots, a_m)$ for both K and K_1 is a forcing condition iff

$$(\exists x_1 \cdots)(\exists y_1 \cdots)\left[P(x_1, \cdots, y_1, \cdots) \wedge \bigwedge_{i \neq j} x_i \neq x_j\right] \in T$$

$(a_1, \cdots, -$ new constants).

Let Γ be the set of formulas $\phi(x_1, \cdots, x_n)$ in L^* such that for any distinct natural numbers $m_1, \cdots, m_n N \models \phi[m_1, \cdots, m_n]$.

We shall prove now

$$(*) \quad K_1^f = T \cup \{\psi(c_{i_1}, \cdots, c_{i_n}) : i_1, \cdots, i_n \text{ are distinct, and } <\omega, \psi(x_1, \cdots, x_n) \in \Gamma\}.$$

CONSTRUCTION. Let A be a countable set of new individual constants, let P be a forcing condition. We shall show that there is a K_1-generic model, which is a model of $K_1^f(P) = \{\phi \in L(A) : P \Vdash^* \phi\}$ and whose reduct to L^* is N. Let $\{\phi_i : i < \omega\}$ be the set of sentences of $L(A)$, $A = \{a_n : n < \omega\}$. We define by induction P_n:

(1) $P_0 = P$.

(2) If P_{3n} is defined, then there is a $Q \supset P_{3n}$, such that $Q \Vdash \phi_n$ or $Q \Vdash \neg \phi_n$. Let $P_{3n+1} = Q$.

(3) If P_{3n+1} is defined it is easy to see that there is a natural number m such that $P_{3n+1} \cup \{c_n = m\}$ is a forcing condition. Let $P_{3n+2} = P_{3n+1} \cup \{c_n = m\}$.

(4) If P_{3n+2} is defined, we can similarly find $P_{3n+3} \supset P_{3n+2}$ such that for some m, $a_n = m \in P_{3n+3}$.

As in [**1**, Theorem 3.3] we get a generic model $N(P)$ which satisfies all our conditions.

Now let us prove (*)

(a) If $\psi \in T$, and not $\varnothing \Vdash^* \psi$ then for some P, $P \Vdash \neg \psi$, so $N(P) \models \neg \psi$. As $\psi \in L^*$, and N is the reduct of $N(P)$ to L^*, $N \models \neg \psi$, contradiction so $T \subset K_1^f$, and as T is complete $K_1^f \cap L^* = T$.

(b) If $\phi(x_1, \cdots, x_n) \in \Gamma$ and not $\varnothing \Vdash^* \phi(c_{i_1}, \cdots, c_{i_n})$ $(i_1, \cdots, i_n$ are distinct) then for some P, $P \Vdash \neg \phi(c_{i_1}, \cdots, c_{i_n})$ so $N(p) \models \neg \phi(c_{i_1}, \cdots, c_{i_n})$, contradiction to the definition of Γ.

(c) Suppose $\phi(c_{i_1}, \cdots, c_{i_n}) \in K_1^f$ $(i_1, \cdots, i_n$ are distinct) (otherwise, we can write ϕ is a different way). So for every distinct natural number $m_1, \cdots, m_n$, $P = \{c_{i_1} = m_1, \cdots, c_{i_n} = m_n\}$ is a forcing condition. So as $\varnothing \Vdash^* \phi(c_{i_1}, \cdots, c_{i_n})$, also $P \Vdash^* \phi(c_{i_1}, \cdots, c_{i_n})$ so $N(P) \models \phi(m_1, \cdots, m_n)$. So $\phi(x_1, \cdots, x_n) \in \Gamma$.

So we prove (*). By [**1**, Theorem 6.1] (and here it can be seen directly)

$$K^f = T \cup \{\psi(c_{i_1}, \cdots, c_{i_n}): i_1, \cdots, i_n < \aleph_1 \text{ are distinct, and } \psi(x_1, \cdots, x_n) \in \Gamma\};$$

clearly, by the definition of Γ, for $i \neq j$, $c_i \neq c_j \in K^f$. So let M be a K-generic model. So it is a model of K^f [**1**, Definitions 3.1, 3.2] so $\|M\| \geqq |\{c_i : i < \aleph_1\}| = \aleph_1 > \aleph_0$. Also M is model complete for K^f hence for T (by the definition of Γ). This contradicts Rabin [**3**], that any nonstandard model of T has an extension which is a model of T but not an elementary extension of M.

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