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PÓLYA PEAKS AND THE OSCILLATION OF POSITIVE FUNCTIONS

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ABSTRACT. A new proof is given of the existence of Pólya peaks of an (increasing) function $g(t)$. This approach yields several applications, including a characterization of those p for which g can have peaks of order p.

Introduction. Many results in function theory depend upon growth properties of increasing functions. For example, the proof of Nevanlinna's second fundamental theorem ultimately depends on some form of a simple but elegant growth lemma due to Borel [**11**, p. 38]. In recent years, much progress has come from a systematic exploitation of the "Pólya peaks" of an increasing function of finite lower order.

If g is a positive increasing function on $[t_0, \infty)$, sequences $r_n, s_n \to \infty$ such that

$$\frac{g(t)}{g(r_n)} \leqq \left(\frac{t}{r_n}\right)^p \{1 + \delta_n\} \qquad (a_n^{-1} r_n \leqq t \leqq a_n r_n), \tag{1}$$

$$\frac{g(t)}{g(s_n)} \geqq \left(\frac{t}{s_n}\right)^p \{1 - \delta_n\} \qquad (a_n^{-1} s_n \leqq t \leqq a_n s_n) \tag{2}$$

hold for some $a_n \to \infty$, $\delta_n \to 0$ are called Pólya peaks of order p, of the first and second kinds respectively, for g. This definition is due to A. Edrei, who has used these peaks to find a variety of interesting function-theoretic results; see for example [**3**], [**4**], [**5**]. Edrei has shown [**4**] that peaks of the first kind exist for every p in $\mu \leqq p \leqq \rho$ where

$$\rho = \limsup_{t\to\infty} \frac{\log g(t)}{\log t}, \qquad \mu = \liminf_{t\to\infty} \frac{\log g(t)}{\log t} \tag{3}$$

denote the order and lower order of g. When $\rho < \infty$, the existence of peaks of the first kind, order $p = \rho$, can be deduced from a growth lemma of

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Pólya ([**11**, p. 101], [**14**]). In [**15**], Edrei's ideas are adapted to show the existence of peaks of the second kind.

The original methods of Pólya [**14**] show that, for a convolution transform

$$G(x) = \int_0^\infty g(t)k\left(\frac{x}{t}\right)\frac{dt}{t} \qquad (0 < x < \infty), \tag{4}$$

with k nonnegative and sufficiently well behaved near 0 and ∞,

$$\liminf_{t\to\infty} \frac{G(t)}{g(t)} \leqq \hat{k}(\rho) \leqq \limsup_{t\to\infty} \frac{G(t)}{g(t)}\ ; \tag{5}$$

here ρ is given by (3) and $\hat{k}(\rho)=\int_0^\infty k(t)t^{-\rho-1}\,dt$ denotes the Mellin transform of k at ρ. Edrei's methods improve (5) by replacing ρ by any number p for which g has Pólya peaks of order p and $\hat{k}(p)<\infty$. Appropriate choices of g and k (cf. [**14**, pp. 177–180], [**8**], [**4**], and item (iii) below) yield interesting value-distribution properties of entire and meromorphic functions.

It is thus of interest to obtain an intrinsic characterization of those p for which Pólya peaks of order p exist, one more manageable than that implied by (1) and (2). Consider

$$\wp_1 = \{p\colon (1) \text{ holds for some } a_n \to \infty,\ \delta_n \to 0\},$$

set $\mu_1=\inf\{p \in \wp_1\}$ and $\rho_1=\sup\{p \in \wp_1\}$, and define μ_2 and ρ_2 similarly in terms of peaks of second kind. Theorem 1 relates these notions to simple oscillation properties of g itself.

THEOREM 1. *Let g be positive on $[t_0, \infty)$, and such that*

$$g(x) > \delta g(t)(x/t)^K \qquad (t_0 \leqq t < x) \tag{6}$$

holds for some $K>-\infty$ and $\delta>0$. Define

$$\rho_* = \sup\left\{p\colon \limsup_{x,A\to\infty} \frac{g(Ax)}{A^p g(x)} = \infty\right\}, \tag{7}$$

$$\mu_* = \inf\left\{p\colon \liminf_{x,A\to\infty} \frac{g(Ax)}{A^p g(x)} = 0\right\}. \tag{8}$$

Then g has Pólya peaks of first and second kind of order p $(<\infty)$ if and only if

$$\mu_* \leqq p \leqq \rho_*. \tag{9}$$

Thus $\rho_1=\rho_2=\rho_*$, $\mu_1=\mu_2=\mu_*$. It is obvious directly from (7) and (8) that μ and ρ, defined in (3), satisfy

$$K \leqq \mu_* \leqq \mu \leqq \rho \leqq \rho_* \leqq \infty \tag{10}$$

where K is the constant in (6). Simple examples show that any combination of equality and inequality in (10) is possible.

The weak tauberian condition (6) is equivalent to the one used in [**1**] and [**2**]. Although in many applications of Pólya peaks one considers only increasing g, our methods apply just as well to g satisfying (6); that some condition like (6) is necessary in Theorem 1 may be seen e.g. from $g(t)=\exp(t \sin t)$.

At the end of this paper, Theorem 1 is applied to give a new proof of a result of Edrei and Fuchs. For the present, we record that the definitions (7), (8) of μ_* and ρ_* are relevant in several other contexts:

(i) When $\rho_* < \infty$, it follows directly from (6) and (7) that

$$g(x) < Cg(t)(x/t)^{\beta} \qquad (\rho_* < \beta, C = C(\beta); t_0 \leqq t < x). \tag{11}$$

Inequalities like (11) are often needed in tauberian problems to estimate the "tails" of convolutions (4); see for instance [**2**, §§ 4 and 5] where these ideas are decisive, and [**9**, §8].

(ii) When g is positive, measurable and regularly varying (in Karamata's sense) so that for each fixed $A>0$ $\lim_{x\to\infty} g(Ax)/g(x)=A^{\rho}$ for some ρ, the estimate

$$A^{\rho-\varepsilon} < g(Ax)/g(x) < A^{\rho+\varepsilon} \qquad (\varepsilon > 0; x \geqq x_0, A \geqq A_0) \tag{12}$$

is classical and useful ([**9**], [**13**]). A glance at our definitions (7) and (8) shows that the class of positive functions with $\mu_* = \rho_* = \rho$ is precisely the class for which (12) holds.

(iii) Finally, we give a typical application to function theory (compare Theorem 1 of [**5**]). Let $f(z)$ be an entire function normalized by $f(0)=1$, and let $M(t,f)=\max_{\theta}|f(te^{i\theta})|$, $L(t,f)=\min_{\theta}|f(te^{i\theta})|$. The method of Denjoy-Kjellberg [**12**] shows (implicitly on pp. 193–196 of [**12**]) that if $0<\alpha<1$, there exist positive constants $k(\alpha)$ and $C(\alpha)$ with

$$\begin{aligned} &\int_r^R \{\log L(t,f) - \cos \pi\alpha \log M(t,f)\} \frac{dt}{t^{1+\alpha}} \\ &\qquad \geqq k(\alpha) \frac{\log M(r,f)}{r^{\alpha}} - C(\alpha) \frac{\log M(4R,f)}{R^{\alpha}} \end{aligned} \tag{13}$$

when $0<r<R$. Thus, if μ_* [defined in (8) with $g(t)=\log M(t,f)$] is less than one, the left side of (13) is positive for $\alpha \in (\mu_*, 1)$ for arbitrarily large r, R/r and so

$$\limsup_{t\to\infty} \frac{\log L(t,f)}{\log M(t,f)} \geqq \cos \pi\mu_*.$$

This is a sharpened form of the classical "cos $\pi\rho$ theorem" which does not involve the order or lower order of f.

1. **One-sided peaks.** Our existence theorem for Pólya peaks will be a simple consequence of

THEOREM 1a. *Let g be positive and continuous on $[t_0, \infty)$, and such that (6) holds there. Let $x_n, A_n \to \infty$ be given, and define p_n by*

$$g(A_n x_n)/g(x_n) = A_n^{p_n} \qquad (n \geqq 1). \tag{1.1}$$

Then there exist sequences $r_n, r_n', s_n, s_n', a_n$ all tending to ∞ and $\varepsilon_n, \delta_n \downarrow 0$ such that

$$\frac{g(t)}{g(r_n)} \leqq \{1+\delta_n\}\left(\frac{t}{r_n}\right)^{p_n(1-\varepsilon_n)} \qquad (a_n^{-1} r_n \leqq t \leqq r_n), \tag{1.2}$$

$$\frac{g(t)}{g(r_n')} \leqq \{1+\delta_n\}\left(\frac{t}{r_n'}\right)^{p_n(1+\varepsilon_n)} \qquad (r_n' \leqq t \leqq a_n r_n'), \tag{1.3}$$

$$\frac{g(t)}{g(s_n)} \geqq \{1-\delta_n\}\left(\frac{t}{s_n}\right)^{p_n(1+\varepsilon_n)} \qquad (a_n^{-1} s_n \leqq t \leqq s_n), \tag{1.4}$$

$$\frac{g(t)}{g(s_n')} \geqq \{1-\delta_n\}\left(\frac{t}{s_n'}\right)^{p_n(1-\varepsilon_n)} \qquad (s_n' \leqq t \leqq a_n s_n'). \tag{1.5}$$

PROOF. Assume first that (6) holds with

$$K = 1, \tag{1.6}$$

so that

$$\liminf_{n\to\infty} p_n \geqq 1. \tag{1.7}$$

Choose $\varepsilon_n < 1$, $\varepsilon_n \downarrow 0$ so slowly that $A_n^{\varepsilon_n} \to \infty$. Define

$$r_n = \inf\left\{t \in [x_n, A_n x_n] : \frac{g(t)}{g(A_n x_n)} \geqq \left(\frac{t}{A_n x_n}\right)^{p_n(1-\varepsilon_n)}\right\}; \tag{1.8}$$

by (1.1), $r_n > x_n$. The point of this choice of r_n is explained by the following estimate: for $x_n \leqq t \leqq r_n$,

$$\begin{aligned} \frac{g(t)}{g(r_n)} &= \frac{g(t)}{g(A_n x_n)}\left(\frac{A_n x_n}{r_n}\right)^{p_n(1-\varepsilon_n)} \\ &\leqq \left(\frac{t}{A_n x_n}\right)^{p_n(1-\varepsilon_n)}\left(\frac{A_n x_n}{r_n}\right)^{p_n(1-\varepsilon_n)} = \left(\frac{t}{r_n}\right)^{p_n(1-\varepsilon_n)}. \end{aligned} \tag{1.9}$$

Thus if $r_n/x_n \to \infty$, we can set $a_n = r_n/x_n$ and (1.2) follows. When $\liminf\{r_n/x_n\} < \infty$, we deduce from (1.8) and (1.1) that

$$\begin{aligned} \frac{g(r_n)}{g(x_n)} &= \frac{g(A_n x_n)}{g(x_n)} \frac{g(r_n)}{g(A_n x_n)} \\ &= A_n^{p_n}\left(\frac{r_n}{A_n x_n}\right)^{p_n(1-\varepsilon_n)} \\ &= A_n^{\varepsilon_n p_n}\left(\frac{r_n}{x_n}\right)^{p_n(1-\varepsilon_n)}, \end{aligned} \tag{1.10}$$

and thus

$$g(r_n)/g(x_n) > A_n^{\varepsilon_n p_n}. \tag{1.11}$$

Now choose y_n ($<x_n$ for all large n, by (1.7)) so that

$$(r_n/y_n)^{p_n(1-\varepsilon_n)} = g(r_n)/g(x_n). \tag{1.12}$$

Thus (1.11) and (1.7) imply

$$r_n/y_n > A_n^{\varepsilon_n} \to \infty, \tag{1.13}$$

and by (6), (1.6) and (1.12)

$$\frac{g(t)}{g(r_n)} < \frac{g(x_n)}{\delta g(r_n)} = \frac{1}{\delta}\left(\frac{y_n}{r_n}\right)^{p_n(1-\varepsilon_n)} \qquad (y_n \leqq t \leqq x_n). \tag{1.14}$$

Putting $z_n = \min(x_n, \delta^{-1/p_n(1-\varepsilon_n)} y_n)$, we have from (1.13) and (1.7) that $r_n/z_n \to \infty$ ($n \to \infty$), and from (1.14) and (1.9) that

$$g(t)/g(r_n) \leqq (t/r_n)^{p_n(1-\varepsilon_n)} \qquad (z_n \leqq t \leqq r_n).$$

This proves (1.2), with $a_n = r_n/z_n$ and $\delta_n = 0$.

The proof of (1.3) is easier; define

$$r_n' = \sup\left\{t \in [x_n, A_n x_n] : \frac{g(t)}{g(x_n)} \geqq \left(\frac{t}{x_n}\right)^{p_n(1+\varepsilon_n)}\right\}. \tag{1.15}$$

Then we have from (6), (1.6), (1.1) and (1.15) that

$$\delta < \frac{g(A_n x_n)}{g(r_n')} = A_n^{p_n}\left(\frac{x_n}{r_n'}\right)^{p_n(1+\varepsilon_n)} = \left(\frac{A_n x_n}{r_n'}\right)^{p_n(1+\varepsilon_n)} A_n^{-\varepsilon_n p_n},$$

so that by (1.7),

$$A_n x_n/r_n' \geqq \eta A_n^{\varepsilon_n/(1+\varepsilon_n)} \to \infty \qquad (n \to \infty) \tag{1.16}$$

for some $\eta>0$. Also, on $r_n'\leqq t\leqq A_nx_n$ we have

$$(1.17)\qquad g(t)/g(r_n') = g(t)/g(x_n)(x_n/r_n')^{p_n(1+\varepsilon_n)} \leqq (t/r_n')^{p_n(1+\varepsilon_n)},$$

and thus (1.3) is a consequence of (1.16) and (1.17).

To verify (1.5) one defines, analogously to (1.8),

$$(1.18)\qquad s_n' = \sup\left\{t \in [x_n, A_nx_n]: \frac{g(t)}{g(x_n)} \leqq \left(\frac{t}{x_n}\right)^{p_n(1-\varepsilon_n)}\right\},$$

and proceeds as in the proof of (1.2). Finally, using

$$s_n = \inf\left\{t \in [x_n, A_nx_n]: \frac{g(t)}{g(A_nx_n)} \leqq \left(\frac{t}{A_nx_n}\right)^{p_n(1+\varepsilon_n)}\right\},$$

(1.4) can be obtained in the same manner as was (1.3).

When g fails to satisfy (1.6), define $h(t)=g(t)t^{1-K}$. Then the hypotheses of Theorem 1a as well as (1.6) are satisfied by h, and the above proof yields (1.2)—(1.5) with g replaced by h, p_n replaced by $q_n\equiv p_n+1-K$, and $\delta_n\equiv 0$. Decreasing the a_n if necessary to achieve $a_n^{\varepsilon_n}=1+o(1)$ $(n\to\infty)$, the conclusion of Theorem 1a follows easily.

2. **Proof of Theorem 1.** From (1), (2) we find that there exist A_n, B_n, x_n, $y_n\to\infty$ such that

$$\frac{g(A_nx_n)}{A_n^pg(x_n)} \leqq 1 + o(1) \leqq \frac{g(B_ny_n)}{B_n^pg(y_n)} \qquad (n \to \infty).$$

(Consider r_n, a_n, $a_n^{-1}r_n$ if (1) holds, for instance.) Hence (9) is necessary for (1) and (2).

For sufficiency, let p $(<\infty)$ satisfy (9), and assume first that g is continuous. Choose A_n, $x_n\to\infty$ so that the p_n in (1.1) satisfy $p_n\to\rho_*$ $(n\to\infty)$, and choose B_n, $y_n\to\infty$ so that the q_n defined by $g(B_ny_n)/g(y_n)=B_n^{q_n}$ satisfy $q_n\to\mu_*$ $(n\to\infty)$. Then there exist $\eta_n\downarrow 0$ such that

$$(2.1)\qquad p - \eta_n < p_n, \qquad q_n < p + \eta_n \qquad (n \geqq 1).$$

Using the first inequality of (2.1) in (1.2), and decreasing the a_n if necessary to achieve $a_n^{\eta_n+\varepsilon_n}\to 1$, we find $r_n^*\to\infty$ such that

$$(2.2)\qquad \begin{aligned}\frac{g(t)}{g(r_n^*)} &\leqq \{1 + \delta_n\}\left(\frac{t}{r_n^*}\right)^{(p-\eta_n)(1-\varepsilon_n)} \\ &= \left(\frac{t}{r_n^*}\right)^p\{1 + o(1)\} \qquad (a_n^{-1}r_n^* \leqq t \leqq r_n^*).\end{aligned}$$

The second part of (2.1) used in (1.3) yields $r'_n \to \infty$ satisfying

$$\begin{aligned}\frac{g(t)}{g(r'_n)} &\leqq \{1+\delta_n\}\left(\frac{t}{r'_n}\right)^{(p+\eta_n)(1+\varepsilon_n)} \\ &= \left(\frac{t}{r'_n}\right)^{p}\{1+o(1)\} \qquad (r'_n \leqq t \leqq a_n r'_n).\end{aligned} \tag{2.3}$$

By passing to subsequences if necessary we may assume that $r^*_n < r'_n$, as well as (2.2) and (2.3), holds for all $n \geqq 1$. Choosing $r_n \in [r^*_n, r'_n]$ by

$$\frac{g(r_n)}{r_n^p} = \sup_{r^*_n \leqq t \leqq r'_n} \frac{g(t)}{t^p}, \tag{2.4}$$

we have (1) when g is continuous. For the general case, put $t_n = t_0 + n$ and consider the step function

$$g_0(x) = \sup_{t_n \leqq t \leqq t_{n+1}} g(t) \qquad (t_n < x < t_{n+1}),$$
$$g_0(t_n) = \sup\{g_0(t_n+), g_0(t_n-)\} \qquad (n \geqq 1).$$

It is not difficult to approximate g_0 by a continuous function $g_1 \geqq g_0$ satisfying (6) and such that the values of ρ_*, μ_* in (7), (8) are not altered when g is replaced by g_1. Further, if $r_n \to \infty$ satisfies (1) with g_1 in place of g, then there exist $R_n \in [r_n - 1, r_n + 1]$ such that $g(R_n) \sim g_1(R_n) \sim g_1(r_n)$ $(n \to \infty)$, and thus g satisfies (1) with r_n replaced by R_n.

The proof of (2) uses (1.4) and (1.5) in an analogous manner.

3. **An application.** Theorem 1 allows a new interpretation, as well as a short proof, of a result of Edrei and Fuchs ([**6**, Theorem 5], [**7**, Lemma A]). Let $f(z)$ be entire, of lower order $\mu < \infty$ and order ρ $(\leqq \infty)$ and put

$$\mathscr{K} = \mathscr{K}(f) = \limsup_{r \to \infty} (N(r,0)/T(r,f)). \tag{3.1}$$

Edrei and Fuchs have shown ([**3**, Theorem 4a] [**10**, Lemma 9.2]) that, for any integer $q \geqq 0$,

$$\begin{aligned}(2-\mathscr{K}-\varepsilon)g(r) &< (\mathscr{K}+\varepsilon)A(q+1)\int_s^R g(t)\left(\frac{r}{t}\right)^q k\left(\frac{r}{t}\right)\frac{dt}{t} \\ &\quad + B\left(\frac{s}{r}\right)^q g(2s) + B\left(\frac{r}{R}\right)^{q+1} g(2R) \qquad (s_0(\varepsilon) \leqq s \leqq \tfrac{1}{2}r \leqq \tfrac{1}{4}R)\end{aligned}$$

where $g(t) = T(t,f)$, $\varepsilon > 0$ and A, B denote absolute constants. Using (1) and an estimate [**6**, p. 302] for $\hat{k}(p-q)$, we obtain

$$\mathscr{K}(f) \geqq \eta \sup_{\mu_* \leqq p \leqq \rho_*} \frac{|\sin \pi p|}{p} \qquad (\eta = \text{absolute constant}).$$

Thus if $\mathscr{K}(f)<\delta$ for small enough δ, the fact that peaks exist throughout the entire interval $[\mu_*, \rho_*]$ yields

$$|\mu_* - n| < \tfrac{1}{4}, \qquad \rho_* - \mu_* < \tfrac{1}{4},$$

say, for some positive integer n.

Since g does not have Pólya peaks of order $n+\frac{1}{2}$, Theorem 1 implies that there exist x_0, A with $g(u)u^{-n-1/2}\leqq g(x)x^{-n-1/2}$ for all u and x which satisfy $x_0<x\leqq u/A$. For any $x>x_0$, choose $t\in[x, Ax]$ with $g(t)t^{-n-1/2}=\sup_{x\leqq u\leqq Ax} g(u)u^{-n-1/2}$, and it is then clear that each interval $[x, Ax]$ has a point t with

$$g(u)u^{-n-1/2} \leqq g(t)t^{-n-1/2} \qquad (t \leqq u < \infty).$$

This is essentially part II of Theorem 5 in [**6**], and part I may be obtained similarly.

ADDED IN PROOF. In a recent paper (*Properties of Pólya peaks*, Rocky Mt. J. Math. **1** (1971), 649–656), H. Silverman states a necessary condition for the existence of Pólya peaks of order p, namely: $\kappa\leqq p\leqq\Omega$, where κ and Ω are obtained as certain iterated limits (loc. cit., p. 651). It is not difficult to see that in fact $\kappa=\mu_*$ and $\Omega=\rho_*$; thus, by Theorem 1 above, Silverman's condition is sufficient as well as necessary.

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